FINITE AND INFINITE DIMENSIONAL HOLOMORPHY

by

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TO

MY MOTHER AND FATHER

WITH

PROFOUND RESPECT

25/5/85

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CERTIFICATE

This is to certify that the research work embodied in the present dissertation entitled ''Finite and Infinite Dimensional Holomorphy'' by Mr. Gunamani Deheri, a Ph.D. scholar of this Department, has been carried out under our joint supervision and that it has not been submitted elsewhere for any degree or diploma.

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CONTENTS

Synopsis

As the title suggests, the present dissertation is a study of holomorphic functions on finite dimensional as well as infinite dimensional spaces. There are seven chapters which comprise this piece of work.

Chapter 1 consists of four sections. In the first section some of the basic definitions and results from the theory of locally convex space including Kolmogorov diameters and nuclear spaces have been incorporated. Section 2 is concerned with some elementary results from the theory of sequence spaces including Köthe spaces. Section 3 deals with some of the definitions and results from the theory of several complex variables and the last section is based on the basic definitions and some results from infinite dimensional holomorphy. The material of this chapter is frequently used in the development of other chapters.

Chapter 2 contains three sections. The first section is a brief account of history and motivation regarding finite dimensional holomorphy. A chronological account of the development of infinite dimensional holomorphy finds its due place in Section 2. The last section is a brief history of the development of nuclear spaces.

In Chapter 3 we consider an extended class of analytic functions of several complex variables and investigate several topological properties of the same including Schwartz property. For the sake of brevity we consider only the two variables case and attempt to extend the class of analytic functions of two variables in terms of infinite matrices. This new class $\Lambda(P)$ of infinite matrices envelops the space of analytic functions on bi-cylinder considered in [3] as well as the space of entire functions of two variables initiated in [2]. Last two sections are devoted to proving Schwartz property. Where as in Section 3, we find the estimations of Kolmogorov diameters for proving the Schock-Terzioglu criterian for the Schwartz property of $\Lambda(P)$, in Section 4 we prove the Schwartz property by making use of the notion of bi-diametral dimension.

Chapter 4 is in continuation to the study of the matrix space $\delta^{\alpha,\beta}$ initiated in Chapter 3 which, in particular, includes the class of entire functions of two variables. In the beginning of this chapter we find the Köthe dual of $\delta^{\alpha,\beta}$ and characterize its bounded sets relative to the weak topology. Then we proceed to characterize the dual of this class as well as of a subclass of this class, which correspond to subspaces of δ having order zero and type almost equal to one.

Besides we also pay sufficient attention to investigate the structural properties of this subspace.

Chapter 5 deals with a new concept of a dual λ_{α}^{μ} of a sequence space λ where μ is another suitable sequence space and α an arbitrary sequence. This new notion of dual envelops, in particular, the well known α-, β-and γ-duals (cf. [4]) and has some applications presented in the next chapter. Here in this chapter we essentially develop the basic results depending on these duals; for instance we establish results relating to the sequential completeness of a sequence space λ with its $\alpha\mu$-perfectness and characterize boundedness of subsets of λ.

In Chapter 6, corresponding to a suitable sequence space μ, we define a class of weighted holomorphic functions on a Banach space, the weights having been provided by μ and introduce a subclass of this class related to the $\alpha\mu$-dual of a sequence space λ. This class extends sufficiently a class studied earlier by Boland [1]. After equipping with suitable locally convex topologies, we study the usual topological structure of these spaces. We characterize bounded subsets of this class and investigate conditions under which the subspace topology coincides with various other topologies on bounded sets. The last section is dealt with the characterization of relatively compact sets.

The last chapter namely, Chapter 7 is related to the study of a class of holomorphic (indeed, hypoanalytic) mappings defined on open subsets of the dual of a nuclear sequence space. Besides obtaining several results on nuclear sequence spaces, we finally obtain the Schauder basis representation (cf. [5] for relevant notions) of elements of this class with respect to compact open topology.

[1] Boland, P.J.: Some Spaces of Entire and Nuclearly Entire Functions on a Banach Space I; J. Reine Angew Math. 270(1974), 38-60.

[2] Kamthan, P.K.: A Study on Space of Entire Functions of Several Complex Variables; Yokohama Math. Jour. 21(1), (1973), 17-20.

[3] Kamthan, P.K.: and Gupta, M. Analytic Functions on Bicylinder Indian J. Pure Appl. Math. 5(12), (1974), 1119-1126.

[4] ____________ : Sequence Spaces and Series; Marcel Dekker Inc. New York 1981.

[5] ____________ : Theory of Bases and Cones; Pitman, Advanced Publishing Programme, London, Feb. 1985.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter we present in brief the salient features from the theory of locally convex spaces including nuclear spaces and Kolmogoroff's diameters, sequence spaces, functions of several variables and holomorphy. For detailed account of these topics we refer to standard texts, research papers and thesis namely, [102], [106], [119], [139], [152], [191], [209], [251].

1.2 Locally Convex Spaces

Unless otherwise stated, we denote throughout by (X,T) (or (E,T)) a Hausdorff topological vector space (abbreviated TVS) or a locally convex space (l.c. TVS), where the topology T is generated by the family $D_T \equiv D$ of pseudonorms or seminorms. The neighbourhood system consisting of balanced convex and absorbing neighbourhoods at the origin for the locally convex topology T, is denoted by U_X (or U_E). For two subsets A and B of X (or E) the symbol ''$A \prec B$'' means ''A is absorbed by B.''

By a <u>total paranorm</u> (resp. an F-<u>norm</u>) we mean a positive real valued function $||\cdot||$ on X, satisfying the conditions (i) $||x|| = 0$ iff $x = 0$; (ii) $||x+y|| \leq ||x|| + ||y||$; (iii) $||-x|| = ||x||$ (resp. $||\lambda x|| \leq ||x||$, $|\lambda| \leq 1$); and (iv) for a sequence $\{\alpha_n\}$ of scalars with $\alpha_n \to \alpha$ and $\{x_n\} \subset X$ with $||x_n - x|| \to 0$, $||\alpha_n x_n - \alpha x|| \to 0$. If (i) is replaced by $||x|| = 0$ if $x = 0$, then a total paranorm is just a <u>paranorm</u>

<u>Proposition</u> 1.2.1: An F-normed space is always a metrizable TVS and conversely.

We make use of

<u>Proposition</u> 1.2.2: A set B in an l.c. TVS X is bounded if and only if for any sequence $\{x_n\}$ in B and sequence $\{\alpha_n\}$ of scalars with $\alpha_n \to 0$ (resp. $\{\alpha_n\} \in \ell^1$) $\alpha_n x_n \to 0$ as $n \to \infty$ (resp. $\sum_{n=1}^{m} \alpha_n x_n$ is a Cauchy sequence in X)

<u>Definition</u> 1.2.3: A closed subspace M of X is said to be (i) <u>co-dimension</u> $\leq n$, i.e., $\dim (X/M) \leq n$ if and only if $M = \bigcap_{i=1}^{n} \ker f_i$ for $f_1, \ldots, f_n \in X^*$ where $\ker f_i = \{x : f_i(x) = 0\}$ and (ii) <u>complemented</u> on X if there exists a closed subspace N of X such that $X = M \oplus N$

<u>Proposition</u> 1.2.4: A closed subspace M of X is complemented in X if $\dim (X/M) < \infty$ (cf. [233]).

For a dual pair $\langle X, Y \rangle$ of vector spaces X, Y defined over the same field $\mathbb{K}$, the symbol's $\sigma(X,Y)$ and $\beta(X,Y)$

respectively stand for weak and strong topology on X.

Regarding linear maps let us recall that a linear map $A: X \to Y$ is called <u>almost open</u> if $\overline{Au} \in U_Y$, $\forall\, u \in U_X$. Further we have

<u>Proposition</u> 1.2.5: Let X and Y be two l.c. TVS. If the transpose map $A': Y^* \to X^*$ of a continuous linear map $A: X \to Y$ is $\sigma(Y^*, Y)-\sigma(X^*, X)$ continuous, then it is $\beta(Y^*, Y)-\beta(X^*, X)$ continuous.

<u>Nuclear Spaces</u>:

For an l.c. TVS X and $u \in U_X$, let $\hat{X}_u$ be the completion of the normed spaces $(X_u, \hat{p}_u)$ where $X_u = X/\ker p_u$ and $\hat{p}_u(x_u) = p_u(x)$, $x_u = x + \ker p_u$, $x \in X$. If $u, v \in U_X$ with $v \prec u$, the natural canonical continuous mapping $K_u^v: X_v \to X_u$ is defined by $K_u^v(x_v) = x_u$, $x_v \in X_v$; and the continuous surjection $K_u: X \to X_u$ is given by $K_u(x) = x_u$, $x \in X$. The mapping K_u^v can be extended continuously to $\hat{K}_u^v: \hat{X}_v \to \hat{X}_u$.

Let us recall from [139], [209] and [245] the following

<u>Definition</u> 1.2.6: A continuous linear operator T from E to F is said to be <u>nuclear</u> if there exists $\{\lambda_n\} \in \ell^1$, $\{f_n\} \subset E$ and $\{y_n\} \subset F$ with $||f_n|| \le 1$, $||y_n|| \le 1$, $n \ge 1$ such that T can be expressed as

$$T(x) = \sum_{n \ge 1} \lambda_n f_n(x) y_n, \quad x \in E$$

Definition 1.2.7: An l.c. TVS (X,T) is said to be nuclear (resp. Schwartz) if to each $u \in U_X$ there exists $v \in U_X$, $v \prec u$ such that $\hat{K}_u^v : \hat{X}_v \to \hat{X}_u$ is nuclear (resp. precompact, that is, it maps bounded sets into precompact sets)

Proposition 1.2.8: Every nuclear Fréchet Space is Montel and hence reflexive. Every quasi-complete Schwartz space is semi-Montel and so every Frechet Schwartz space is Montel.

Schauder Bases:

We refer to [140] and [245] for the details of the theory of Schauder bases.

Definition 1.2.9: A Schauder base in an l.c. TVS(X,T) is a biorthogonal pair $\{x_n, f_n\}$ of sequences $\{x_n\} \subset X$, $\{f_n\} \subset X^*$ such that

$$x = \sum_{n \geq 1} f_n(x) x_n, \quad x \in X,$$

when the convergence of the infinite series is considered in the topology T.

Definition 1.2.10: A Schauder base $\{x_n, f_n\}$ is said to be equicontinuous, if for each $u \in U_X$, there exists $v \in U_X$ satisfying

$$|f_n(x)| p_u(x_n) \leq p_v(x), \quad x \in E, \ n \geq 1$$

Definition 1.2.11: Corresponding to an arbitrary sequence

space λ (cf. next section), a Schauder base $\{x_n, f_n\}$ for an l.c. TVS (X,T) is called <u>fully</u> λ-<u>base</u> if for each p in D_T and x in X $\{f_n(x)\ p(x_n)\} \in \lambda$ and the mapping $\Psi_p: X \to \lambda$, $\Psi_p(x) = \{f_n(x)p(x_n)\}$ is T-$\eta(\lambda,\lambda^x)$ continuous, where $\eta(\lambda,\lambda^x)$ is the normal topology on λ (cf. Section 3). For $\lambda = \ell^1$, $\{x_n, f_n\}$ is called as <u>absolute</u> <u>base</u>.

<u>Kolmogorov Diameters</u>:

For the subject matter of Kolmogorov diameters we refer to [209], [251]. We mention here in brief, what we need in the subsequent chapters.

<u>Definition</u> 1.2.12: Let A and B be subsets of a vector space X with $A \prec B$. Then the <u>n-th Kolmogorov diameter</u> of A with respect to B is the number $\delta_n(A,B)$ defined as

$$\delta_n(A,B) = \inf\{\delta > 0: A \subset \delta B+F, F \text{ a subspace of } X \text{ of dimension at most } n\}$$

<u>Proposition</u> 1.2.13 (i) If $A \subset \rho B$ for some $\rho \geq 0$, then

$$\rho \geq \delta_o(A,B) \geq \delta_1(A,B) \geq \dots \geq 0$$

(ii) For $\lambda,\mu > 0$,

$$\delta_n(\lambda A, \mu B) = \lambda/\mu\ \delta_n(A,B)$$

(iii) If $A \prec B$ and $B \prec C$, then

$$\delta_{m+n}(A,C) \leq \delta_m(A,B)\ \delta_n(B,C)$$

<u>Proposition</u> 1.2.14: Let X and Y be two vector spaces and A, B subsets of X with $A \prec B$. If T is a linear operator from X to Y, then

$$\delta_n(T(A), T(B)) \leq \delta_n(A, B).$$

<u>Proposition</u> 1.2.15: For two sets A and B in an l.c. TVS (X, T) with $\bar{A} \prec B$, B being balanced and convex

$$\delta_n(A, \bar{B}) = \delta_n(\bar{A}, \bar{B}) \leq \delta_n(A, B) \leq \delta_n(\bar{A}, B)$$

For the next result, let us denote by $H(A, B, n)$, the set of all positive numbers δ satisfying the condition

$$A \subset \delta B + \Gamma\{x_1, \ldots, x_n\}, \; x_1, \ldots, x_n \in X_B$$

where A and B are subsets of an l.c. TVS (X, T) such that $A \prec B$; B is balanced, convex and bounded; and $\Gamma\{x_1, \ldots, x_n\}$ is the balanced convex hull of $\{x_1, \ldots, x_n\}$.

<u>Proposition</u> 1.2.16: $\delta_n(A, B) = \inf\{\delta > 0 : \delta \in H(A, B, n)\}$

Next, we have

<u>Proposition</u> 1.2.17: A bounded subset B of an l.c. TVS (X, T) is precompact if and only if for each $u \in U_X$, $\delta_n(B, u) \to 0$ as $n \to \infty$.

From [251], let us quote

<u>Proposition</u> 1.2.18: $\delta_n(K_u(v), K_u(u)) = \delta_n(u, v)$.

<u>Proposition</u> 1.2.19: Let A be a bounded subset of a normed

space $(E, ||\cdot||)$ and $U = \{x \in E: ||x|| \leq 1\}$. If L is a $(n+1)$ dimensional subspace of E and $\alpha \geq 0$ satisfies the relation

$$\alpha \quad (U \cap L) \subset A,$$

then

$$\delta_n(A, U) \geq \alpha.$$

<u>Proposition</u> 1.2.20: A bounded set A of a normed space $(E, ||.||)$ is contained in a space of dimension less than n if and only if $\delta_n(A, U) = 0$.

1.3 <u>Sequence Space</u>

The purpose of this section is to present some definitions and results from the theory of sequence spaces, which are to be used in the sequel; from [189] and [232].

Let us denote ω and φ respectively the vector spaces of scalar valued sequences $\{a_n\}$ in $\mathbb{K}$ and the span of $\{e^n, n \geq 1\}$, where e^n stands for the sequence $\{0, \ldots, 0, 1, 0, \ldots\}$; known as the n-th unit vector. A
nth-coordinate
<u>sequence space</u> λ is a subspace of ω containing φ. Sometimes, we use the symbols a, b, c to denote the sequences $\{a_n\}$, $\{b_n\}$ $\{c_n\}$ respectively in ω. For $a, b \in \omega$, we define $ab = \{a_n b_n\}$. Further we say, $a \in \omega$ is <u>positive</u> if $a_n \geq 0$, for all n.

The symbols c_o, ℓ^p, $0 < p < \infty$, ℓ^∞ or m cs and bs have their usual meaning (cf. [139], p. 31); these are

respectively the space of null, p-th power summable, bounded, convergent series and bounded partial sum sequences.

The α-or Köthe dual, β-, γ-dual of a sequence space λ are respectively defined as

$$\lambda^x \text{ or } \lambda^\alpha = \{a : a \in \omega,\ ab \in \ell^1,\ \forall\, b \in \lambda\};$$

$$\lambda^\beta = \{a : a \in \omega,\ ab \in cs,\ \forall\, b \in \lambda\}$$

$$\lambda^\gamma = \{a : a \in \omega,\ ab \in bs,\ \forall\, b \in \lambda\}$$

For the sake of convenience, we write

$$\lambda^x_+ = \{b \in \lambda^x : b_n > 0,\ \forall\, n\}$$

For $b \in \lambda^x$, we define the seminorms p_b and q_b as follows:

$$p_b(a) = \sum_{n\geq 1} |a_n b_n|,\ a \in \lambda$$

$$q_b(a) = \left|\sum_{n\geq 1} a_n b_n\right|,\ a \in \lambda$$

The locally convex topology generated by the family $\{p_b : b \in \lambda\}$ (resp. $\{q_b : b \in \lambda^x\}$) is known as the normal (resp. weak) topology on λ and is denoted by $\eta(\lambda, \lambda^x)$ (resp. $\sigma(\lambda, \lambda^x)$).

We also make use of the following from [139]

Definitions 1.3.1: A sequence space λ is said normal (resp. perfect) if $a \in \lambda$ whenever $|a_n| \leq |b_n|$, $n \geq 1$ for some $b \in \lambda$ (resp. $\lambda = \lambda^{xx}$). A seminorm p on λ

is called <u>solid</u> if $p(a) \leq p(b)$ whenever $|a_n| \leq |b_n|$, $\forall\ n \geq 1$. A sequence space λ equipped with a linear topology T is known as a K-<u>space</u> if the co-ordinate maps $P_i : \lambda \to \mathbb{K}$ defined by $P_i(a) = a_i$ are continuous and an element $a \in \lambda$ is said to have the AK-<u>property</u> if the n-th <u>section</u> $a^{(n)}$, where $a^{(n)} = \sum_{i=1}^{n} a_i e^i$ tends to a in the topology T. A K-space λ is known as an AK-<u>space</u> if each of its elements possess the AK-property. By the <u>length</u> of an element a in φ, we mean the subscript of the last non zero co-ordinate of a. Further a subset A of φ is said to be of bounded length ℓ, if for each a in A, $a_i = 0$ for $i \geq \ell + 1$.

<u>Proposition</u> 1.3.2: A sequence space λ is perfect if and only if λ is $\sigma(\lambda,\lambda^x)$-sequentially complete or $\eta(\lambda,\lambda^x)$-sequentially complete or $\eta(\lambda,\lambda^x)$-complete.

<u>Proposition</u> 1.2.2: A subset B of φ is $\sigma(\varphi,\omega)$ bounded if and only if there exists a sequence $\{r_i\} \in \omega$ with $r_i > 0$, $i \geq 1$ such that $|a_i| < r_i$, for each a in B, and B is of bounded length.

<u>Proposition</u> 1.3.4: (i) On ω, $\sigma(\omega,\varphi) = \eta(\omega,\varphi) = \beta(\omega,\varphi)$ and (ii) on φ, $\sigma(\varphi,\omega) \subsetneq \eta(\varphi,\omega) = \beta(\varphi,\omega)$. Further the sequential convergence on $\sigma(\varphi,\omega)$ is same as sequential convergence in $\beta(\varphi,\omega)$.

<u>Köthe Space</u>

In this subsection we mention a particular type of perfect sequence spaces. Indeed, we have

Definition 1.3.5: A subset P of ω satisfying the conditions (i) each $a \in P$ is positive (ii) for a,b in P, there exists c in P with $a_n \le c_n$, $b_n \le c_n$, $\forall$ n and (iii) for each $n \in \mathbb{N}$, there exists $a \in P$ with $a_n > 0$, is called a Köthe set or a power set and the sequence space

$$(1.3.6) \qquad \Lambda(P) = \{b \in \omega : p_a(b) = \sum_{n\geq 1} |b_n| a_n < \infty \ , \ \forall \ a \in P\}$$

is known as a Köthe space generated by P.

The natural locally convex topology on $\Lambda(P)$ generated by the family $\{p_a : a \in P\}$ of seminorms is denoted by T_P. It is known that $(\Lambda(P), T_P)$ is complete (cf. 209 , p. 98)

Definition 1.3.7: A Köthe space $\Lambda(P)$ is said to be a G_∞-space or a smooth sequence space of infinite type [resp. a G_1-space or a smooth sequence space of finite type] provided P satisfies the following additional conditions;

(iv) for each $a \in P$, $0 < a_n \le a_{n+1} \le \ldots$ [resp. $0 < a_{n+1} \le a_n < \ldots$]

(v) for each $a \in P$, there exists $b \in P$ such that $a_n^2 \le b_n$ [resp. $a_n \le b_n^2$], $n \ge 1$. In this case we say briefly that P is a G_∞-[resp. G_1-] Kothe set.

A positive increasing sequence $a = \{a_n\}$ is called an exponent sequence if $\lim_n a_n = \infty$. For such a sequence a, let us write

$$P_1 = \{\{R^{a_n}\} : R > 0\}$$

and

$$P_2 = \{\{R^{a_n}\} : R < 1\}$$

Then one can easily show that P_1 and P_2 are respectively Köthe sets of infinite and finite type. The Köthe space $\Lambda(P_1)$ [resp. $\Lambda(P_2)$] is called a power series space of infinite type [resp. power series space of finite type] and is denoted by $\Lambda_\infty(a)$ [resp. $\Lambda_1(a)$] . For $a_n = \log n$, the space $\Lambda_\infty(a)$ is the well known Fréchet space S of all rapidly decreasing sequences.

Following two results, respectively known as Grothendieck-Pietsch criterion for nuclearty and Schock-Terzioglu criterion for Schwartz-property are taken from [139]

Proposition 1.3.8: A sequence space $(\lambda, \eta(\lambda,\lambda^x))$ is nuclear if and only if for each $a \in \lambda^x_+$, there exists $b \in \lambda^x_+$, $a_n \leq b_n$, $\forall\, n \geq 1$ such that $\{a_n/b_n\} \in \ell^1$. In case $\lambda = \Lambda(P)$, λ^x_+ is replaced by P.

Proposition 1.3.9: A sequence space $(\lambda, \eta(\lambda,\lambda^x))$ is Schwartz if and only if for each $a \in \lambda^x_+$, there exists $b \in \lambda^x_+$, $a_n \leq b_n$, $\forall\, n \geq 1$ such that $\{a_n/b_n\} \in c_0$. For $\lambda = \Lambda(P)$, λ^x_+ is replaced by P.

Köthe Matrix Space

We follow [138] to recall a few unfamiliear definitions and notation. By Ω we mean the collection of all scalar valued infinite matrices (a_{mn}) with usual coordinatewise

vector operations. Let (e^{mn}) denote the infinite matrix whose all entries are zeros except the meet of m-th row and n-th column which is one. Any subspace Λ of Ω with $\Lambda \supset \Phi$, the linear space generated by $\{e^{mn} : m+n \geq 0\}$, is called a <u>Matrix space</u>. For any matrix space Λ, let us introduce the K-<u>dual</u> $\Lambda^{\#}$ of Λ as follows:

$$\Lambda^{\#} = \{b = (b_{mn}) \in \Omega : \sum_{m+n>0}\sum |a_{mn} b_{mn}| < \infty , \forall\, a \in \Lambda \}$$

Further for $a \in \Omega$, if

$$a^{(s)} = \sum_{0 \leq m+n \leq s}\sum a_{mn} e^{mn},$$

then $a^{(s)}$ is called the s-th <u>plane section</u> of the matrix $a = (a_{mn})$.

At the end of this section, let us also recall the following three infinite matrix spaces

$$\ell^{11} = \{a = (a_{mn}) : \sum_{m+n \geq 0}\sum |a_{mn}| < \infty\} ;$$

$$c_{oo} = \{a = (a_{mn}) : a_{mn} \to 0 \text{ as } m+n \to \infty\} ;$$

and

$$\ell^{\infty\infty} = \{a = (a_{mn}) : \sup_{m,n} |a_{mn}| < \infty , m,n \geq 0\}.$$

1.4 Functions of Several Complex Variables

For $n \in \mathbb{N}$, $n \geq 1$, we write $\mathbb{C}^n$, $\mathbb{R}^n$ and $\mathbb{N}^n$, for the cartesian products $\underbrace{\mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C}}_{n\text{-times}}$, $\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n\text{-times}}$

and $\underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{n\text{-times}}$ respectively. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ $\mathbb{C}^n$ or $\mathbb{R}^n$ or $\mathbb{N}^n$, we define

$$||\alpha|| = ||(\alpha_1, \ldots, \alpha_n)|| = |\alpha_1| + \ldots + |\alpha_n|$$

Then the spaces $\mathbb{C}^n$ and $\mathbb{R}^n$ equipped with these norms are Banach spaces where as $\mathbb{N}^n$ is a metric space. We will denote a member of $\mathbb{C}^n$, $\mathbb{R}^n$ and $\mathbb{N}^n$ respectively by z, r and m. For $m \in \mathbb{N}^n$, we set

$$m! = m_1! \ldots m_n!$$

If $a \in \mathbb{C}^n$ and $\rho = (\rho_1, \ldots, \rho_n)$, the polycylinder (polydise) with centre 'a' and (poly) radious 'ρ' is the set

$$D(a,\rho) = \{z \in \mathbb{C}^n : |z_i - a_i| < \rho_i,\ 1 \leq i \leq n\}.$$

$\bar{D}(a,\rho)$ stands for closure of $D(a,\rho)$ in $\mathbb{C}^n$. Further, if $z \in \mathbb{C}^n$ and $m \in \mathbb{N}^n$, we write

$$(1.4.1) \qquad z^m = z_1^{m_1} \ldots z_n^{m_n}$$

Definition 1.4.2: A complex valued function f defined on a region $\Omega \subset \mathbb{C}^n$, is said to be holomorphic on Ω if for each $z \in \Omega$ it has partial derivatives

$$\frac{\partial f}{\partial z_k} = \lim_{\Delta z_k \to 0} \frac{f(z_1, \ldots, z_{k-1}, z_k + \Delta z_k, z_{k+1}, \ldots, z_n) - f(z_1, \ldots, z_n)}{\Delta z_k},$$

$$\forall\, k = 1, \ldots, n$$

The function f is said to be holomorphic at a point $z \in \mathbb{C}^n$, if it is holomorphic in some neighbourhood U_z of z. A function defined on $\mathbb{C}^n$ is said to be entire if it is holomorphic for every $z \in \mathbb{C}^n$.

For details of the theory of functions of several complex variables, we refer to any of the texts [113], [117] and [144]. However, we mention here a few, namely.

Proposition 1.4.3: If a series $\sum_{n\geq 1} f_n(z)$ consisting of functions holomorphic in some region $\Omega \subset \mathbb{C}^n$, converges uniformly in that region, then its sum is holomorphic on that region. Partial derivatives of all orders of f may be obtained by term-wise differentiation of the original series. The series thus obtained, converges uniformly in Ω.

Theorem 1.4.4: If a function f is holomorphic in a polycylinder

$$D(a,R) = \{(z_1,\ldots,z_n) : |z_i - z_i| < R_i;\ 1 \leq i \leq n\},$$

then at all point z of the polycylinder

$$(1.4.5)\quad f(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n\geq 0} C_{k_1,\ldots,k_n}(z_1-a_1)^{k_1}\cdots(z_n-a_n)^{k_n}$$

where

$$(1.4.6)\quad C_{k_1,\ldots,k_n} = \frac{1}{k_1!\ldots k_n!}\left[\frac{\partial^{k_1+\ldots+k_n} f}{\partial z_1^{k_1}\ldots\partial z_n^{k_n}}\right]_{z_i=a_i,\,1\leq i\leq n}$$

The series (1.4.5) converges absolutely and uniformly in the polycylinder $D(a,R)$. The representation of the function $f(z_1,\ldots,z_n)$ by the series (1.4.5) is unique.

Proposition 1.4.7: The series (1.4.5) converges on $D(a,R)$ if and only if

$$(*) \qquad \lim_{\|k\|\to\infty} (|C_k| R^k)^{\frac{1}{\|k\|}} \leq 1$$

where $k=(k_1,\ldots,k_n)$, $R^k = R_1^{k_1}\ldots R_n^{k_n}$ and $\|k\| = k_1+\ldots+k_n$. In case of entire function (*) is replaced by

$$\lim_{\|k\|\to\infty} |C_k|^{\frac{1}{\|k\|}} = 0$$

Theorem 1.4.8: Let $D(a,\rho)$ be an open polydisc and f a function continuous on $\bar{D}(a,\rho)$ and analytic in each z_j. Then

$$f(z_1,\ldots,z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D(a,\rho)} \frac{f(\xi_1,\ldots,\xi_n)}{(\xi_1-z_1)\ldots(\xi_n-z_n)}\, d\xi_1\ldots d\xi_n$$

$$(1.4.9) \qquad = \frac{1}{(2\pi i)^n} \int_{|\xi_1-a_1|=\rho_1} \cdots \int_{|\xi_n-a_n|=\rho_n} \frac{f(\xi_1,\ldots,\xi_n)\, d\xi_1\ldots d\xi_n}{(\xi_1-z_1)\ldots(\xi_n-z_n)}$$

and f is analytic on $D(a,\rho)$.

Proposition 1.4.10 (Cauchy Inequality) Let $f(z)$ be

holomorphic on the polycylinder $D(a,R)$, satisfying the condition $|f(z)| < M$. If f is represented by the series (1.4.5), then

(1.4.11) $$|c_k| \leq \frac{M}{R^k}$$

where $k = (k_1,\ldots,k_n)$ and $R^k = R_1^{k_1}\ldots R_n^{k_n}$.

Lastly, we have from [40]

Definition 1.4.12: For $n \in \mathbb{N}$, we write

$$\mathbb{N}^n = \{m : m=(m_1,\ldots,m_n,0,\ldots),\ m_i \in \mathbb{N},\ i=1,\ldots,n\}$$

and $\mathbb{N}^{(\mathbb{N})} = \bigcup_{n\geq 1} \mathbb{N}^n$. For $m \in \mathbb{N}^{(\mathbb{N})}$ with $m=(m_1,\ldots,m_n,0,\ldots)$ and $z \in \omega$, define a map $f^m : \omega \to \mathbb{C}$ by

$$f^m(z) = z^m = \prod_{n=1}^{\infty} z_n^{m_n}$$

where z^m is the one as given in (1.4.1). Then the mappings $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ or $\{z^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ are called monomials.

1.5 Holomorphy

For the results and definitions of this section, the references are [36], [102] and [191].

Holomorphic Mappings

For two Banach spaces E and F, we denote respectively by $\mathcal{L}_a({}^mE,F)$ and $\mathcal{L}_{as}({}^mE,F)$ the vector space of

all m-linear mappings and a subspace of $\mathcal{L}_a({}^mE,F)$ consisting of symmetric mappings. The subspaces of $\mathcal{L}_a({}^mE,F)$ consisting of continuous m-linear mappings and continuous symmetric m-linear mappings are denoted by $\mathcal{L}({}^mE,F)$ and $\mathcal{L}_s({}^mE,F)$ respectively. The space $\mathcal{L}({}^mE,F)$ is a Banach space with respect to the norm

$$||A|| = \sup_{x_1\neq\cdots\neq x_m\neq 0} \frac{||A(x_1,\ldots,x_m)||}{||x_1||\cdots||x_m||}$$

and $\mathcal{L}_s({}^mE,F)$ is a closed subspace of $\mathcal{L}({}^mE,F)$.

For $m=0$, we set

$$\mathcal{L}_a({}^oE,F) = \mathcal{L}_{as}({}^oE,F) = \mathcal{L}({}^oE,F) = \mathcal{L}_s({}^oE,F) = F$$

<u>Definition</u> 1.5.1: To each $A \in \mathcal{L}_a({}^mE,F)$ and $x \in E$, we write

$$Ax^m = \underbrace{A(x,\ldots,x)}_{m\text{-times}} \quad , m \geq 1$$

and $Ax^o=A$ if $m=0$. The mapping $P_m: E \rightarrow F$, defined corresponding to $A \in \mathcal{L}_a({}^mE,F)$ by the relation

$$P_m(x) = Ax^m, \; x \in E$$

is known as the m-<u>homogeneous</u> <u>polynomial</u> associated with A. The vector space of all m-homogeneous polynomials from E to F, associated with all $A \in \mathcal{L}({}^mE,F)$, is denoted by $\mathcal{P}_a({}^mE,F)$ and the subspace consisting of all continuous

m-homogeneous polynomials is designated as $\mathcal{P}({}^mE,F)$ which is a Banach space with the norm, given by

$$\|P_m\| = \sup_{x\neq 0} \frac{\|P_m(x)\|}{\|x\|^m}$$

Next, we have

Definition 1.5.2: A continuous polynomial P from E to F is a mapping from E to F given by

(*) $$P = P_0+\cdots+P_m$$

where $P_k \in \mathcal{P}({}^kE,F)$, $k=0,\ldots,m$, for some $m \in \mathbb{N}$. The symbol $\mathcal{P}(E,F)$ stands for the vector space of all such polynomials from E to F. The representation of each P in $\mathcal{P}(E,F)$ given by (*) is unique.

Let us now consider some special type of continuous polynomials. To begin with, let us observe that $g^m \in \mathcal{P}({}^mE,\mathbb{C}) \equiv \mathcal{P}({}^mE)$, for $g \in E^*$. This leads to

Definition 1.5.3: A member of the subspace of $\mathcal{P}({}^mE)$, spanned by the collection $\{g^m : g \in E^*\}$ is called a polynomial of finite type and the collection of such polynomials is denoted by $\mathcal{P}_f({}^mE)$.

On the subspace $\mathcal{P}_f({}^mE)$ of $\mathcal{P}({}^mE)$, we have another stronger norm $\|\cdot\|_N$ known as nuclear norm defined by

(1.5.4) $$\|P\|_N = \inf\{\sum_{i=1}^{n} \|\varphi_i\|^m : P = \sum_{i=1}^{n} \varphi_i^m,\ \varphi_i \in E^*, i=1,\ldots,n\}$$

where the infimum is taken over all possible representation of P.

The <u>completion</u> of $(\mathcal{P}_f(^mE), ||\cdot||_N)$ in $\mathcal{P}(^mE)$ is the Banach space $\mathcal{P}_N(^mE)$ whose members are called <u>nuclear</u> m-<u>homogeneous</u> polynomials on E.

Concerning power series, we have

<u>Definition</u> 1.5.5: A <u>power series</u> from E to F about $\xi \in E$ is a series in $x \in E$ of the form

$$(*) \qquad \sum_{m \geq 0} P_m(x-\xi)$$

where $P_m \in \mathcal{P}(^mE,F)$, $m=0,1,\ldots$

The largest number r, $0 \leq r \leq \infty$, for which the power series given in (*) converges uniformly on each $\bar{B}_\rho(\xi)$, $0 \leq \rho < r$, where $B_\rho(\xi)$ is the open ball of radious ρ having centre at ξ, is known as the <u>radious of convergence</u>. It is given by

$$r = \frac{1}{\overline{\lim}\ ||P_m||^{1/m}}$$

Further, we have

<u>Proposition</u> 1.5.6: A power series (*) is convergent if and only if the sequence $\{||P_m||^{1/m}, m \geq 1\}$ is bounded. Also, if the sum of (*) is equal to zero for each $x \in \bar{B}_\rho(\xi)$ for some $\rho > 0$, then $P_m=0$ for each m.

We now begin our discussion on holomorphic mappings from the following

Definition 1.5.7: Let U be a nonempty open subset of E. A mapping $f: U \to F$ is said to be holomorphic on U if, corresponding to every $\xi \in U$, there exist power series $\sum_{m\geq 0} P_{m,\xi}(x-\xi)$ from E to F about ξ and some $\rho > 0$ such that $B_\rho(\xi) \subset U$ and

$$(1.5.8) \qquad f(x) = \sum_{m=0}^{\infty} P_{m,\xi}(x-\xi)$$

uniformly for $x \in B_\rho(\xi)$. In this case the power series is usually known as the Taylor series of f at ξ. The collection of all holomorphic mappings from U to F is a vector space with respect to usual pointwise vector operations and is denoted by $H(U,F)$.

For $f \in H(U,F)$ having representation (1.5.8), we define the m-differentials of f at ξ by

$$(1.5.9) \qquad \hat{d}^m f(\xi) = m! P_{m,\xi}$$

where $\hat{d}^m f$ maps U into $\mathcal{P}(^mE,F)$.

Remark: The above definition of holomorphic function can be given on an open subset U of a TVS when F is sequentially complete with appropriate extensions of the definition of m-homogeneous polynomials etc. Some more extensions of the notion of holomorphic mappings are contained in

Definition 1.5.10: Let E,F be l.c. TVS and U a non-void

open subset of E. Then f is said to be Silva holomorphic on U, if for every absolutely convex bounded set B of E and every absolutely convex neighbourhood v of 0 in F, the restriction of $K_V o f$ to $U \cap E_B$ with range in the normed space F_V is holomorphic, where E_B is the normed space $\bigcup\limits_{n \geq 1} nB$ normed by the gauge of B and F_V is the usual quotient space $F/\ker p_V$.

Definition 1.5.11: Let E and F be two TVS, F being sequentially complete. A mapping $f: U \to F$ is said to be G-analytic on U, if for every $\xi \in U$, there are $P_{m,\xi} \in \mathcal{P}_a({}^mE,F)$ such that

$$(1.5.12) \qquad f(\xi + h) = \sum_{m=0}^{\infty} P_{m,\xi}(h)$$

where the series converges uniformly for all h in a neighbourhood of origin in E.

Topologies on $H(U,F)$

On the space $H(U,F)$, we have three natural locally convex topologies defined in

(1.5.13) Compact Open Topology τ_0: For a compact subset K of U and $f \in H(U,F)$, we define a seminorm

$$p_K(f) = \sup_{x \in K} \|f(x)\|$$

The topology generated by the family of all such seminorms

$\{p_K : K \subset U\}$ is called the <u>compact open topology</u> or <u>the topology of uniform convergence on compact sets</u> and is <u>denoted</u> by τ_o.

(1.5.14) <u>Nachbin Topology</u> τ_ω: A seminorm p on H(U,F) is said to be <u>ported</u> by a compact subset K of U if, given any open subset v of U containing K, there exists a constant $C \equiv C(v) > 0$ for which

$$p(f) \leq C \sup_{x \in v} \|f(x)\|$$

holds for every $f \in H(U,F)$. The topology generated by all seminorms on H(U,F), which are ported by compact subsets of U is the <u>Nachbin's topology</u> denoted by τ_ω.

(1.5.15) <u>Countable Compact Open Topology</u> τ_δ. It is the topology generated by the family of all those seminorms p on H(U,F), which satisfy the condition; for every increasing countable open cover $\{U_n\}$ of U, there exists a constant $C > 0$ and $n \in \mathbb{N}$ such that

$$p(f) \leq C \sup_{x \in U_n} \|f(x)\| \text{ , } \forall f \in H(U,F)$$

In general, $\tau_o \subset \tau_\omega \subset \tau_\delta$. But,

$$\tau_o = \tau_\omega = \tau_\delta \iff \dim E < \infty \text{ .}$$

The bornological topologies associated with τ_o and τ_ω are respectively denoted by $\tau_{o,b}$ and $\tau_{\omega,b}$. For the details of these topologies we refer to [64] and [190].

<u>Several Notions in Holomorphy</u>

To begin with, we recall from [191] the holomorphy type contained in

<u>Definition</u> 1.5.16: A <u>holomorphy type</u> θ from E to F is a sequence of Banach spaces $(\mathcal{P}_\theta({}^mE,F), \|\cdot\|_\theta)$, $m \in \mathbb{N}$ satisfying the following conditions:

(i) Each $\mathcal{P}_\theta({}^mE,F)$ is a subspace of $\mathcal{P}({}^mE,F)$;

(ii) $\mathcal{P}_\theta({}^oE,F)$ coincides with $\mathcal{P}({}^oE,F)$ as a normed vector space;

(iii) There exists a real number $\sigma \geq 1$ such that for given $n \in \mathbb{N}$, $m \in \mathbb{N}$, $n \leq m$, $x \in E$ and $P \in \mathcal{P}_\theta({}^mE,F)$,

$$\hat{d}^n P(x) \in \mathcal{P}_\theta({}^nE,F)$$

and

$$\left\| \frac{1}{n!}\hat{d}^n P(x)\right\|_\theta \leq \sigma^m \|P\|_\theta \|x\|^{m-n}$$

<u>Definition</u> 1.5.17: A given $f \in H(U,F)$ is said to be of θ-<u>holomorphy type</u> at $\xi \in U$, if

(i) $\hat{d}^m f(\xi) \in \mathcal{P}_\theta({}^mE,F)$ for $m \in \mathbb{N}$

(ii) There are real numbers $K \geq 0$ and $C > 0$ such that

$$\left\| \frac{1}{m!}\hat{d}^m f(\xi)\right\|_\theta \leq KC^m \quad \text{for } m \in \mathbb{N}$$

Moreover, f is said to be of θ-<u>holomorphy type on U</u> if f is of θ-holomorphy type at every point of U.

Remark: The above notions are useful in unifying the results on compact and nuclear holomorphic mappings.

Following is taken from [47].

Definition 1.5.18: Let E and F be two Banach spaces and $H(K,F)$ be the collection of all F-valued mappings which are holomorphic on some open subsets of E containing K. Then two mappings f_1 and f_2 in $H(K,F)$, defined on open subsets U_1 and U_2, respectively, are said to be equivalent modulo K, if there is an open subset U of E containing K and contained in $U_1 \cap U_2$ such that $f_1(x) = f_2(x)$ for every $x \in U$. Each equivalence class defined with the help of this relation is referred to as a holomorphic germ on K.

Next, we recall from [59]

Definition 1.5.19: An open subset U of a Banach space E is said to be Runge set if the polynomials on E are dense in $H(U)$ with respect to the compact open topology.

Lastly, we quote from [36], [102] the following

Definition 1.5.20: Let E be an l.c. TVS. For $T \in (H(E), \tau_0)^*$, the Borel transform $\hat{T} \in (H(E^*), \tau_0)$ of T is given by

$$\hat{T}(\varphi) = T(e^{\varphi})$$

where $\varphi \in E^*$ and $e^{\varphi} = \sum_{n \geq 0} \frac{\varphi^n}{n!} \in H(E)$.

Definition 1.5.21: Let E_β^* be strong dual of an l.c. TVS E and $f \in H(E_\beta^*)$. Then f is said to be of exponential type on E_β^* if there exists a balanced, convex and bounded subset B of E and a constant $C > 0$ such that

$$|f(\Psi)| \leq C\, e^{p_{B^{o}}(\Psi)} \quad , \forall \Psi \in E^*$$

where $p_{B^{o}}(\Psi) = \sup \{|\Psi(x)| : x \in B\}$. We write
$\mathrm{Exp}\, E_{\beta}^{*} = \{f : f \in H(E_{\beta}^{*}),\ f$ is of exponential type on $E_{\beta}^{*}\}$.

<u>Definition</u> 1.5.22: Let E be a Banach space. An <u>exponential nuclear polynomial</u> on E is a polynomial of the form Pe^{φ} where $\varphi \in E^*$ and $P \in H_{Nb}(E)$, the space of all holomorphic mappings $f : E \to \mathbb{C}$ such that $\hat{d}^n f(0) \in \mathcal{P}_N({}^nE)$ and

$$\lim_{n \to \infty} \left(\frac{\|\hat{d}^n f(0)\|_N}{n!}\right)^{1/n} = 0$$

<u>Definition</u> 1.5.23: Let E be an l.c. TVS and τ a locally convex topology on $H(E)$. Then a <u>convolution operator</u> Q on $(H(E),\tau)$ is a continuous linear mapping $Q : H(E) \to H(E)$ which commutes with its translations. A convolution operator Q is said to satisfy <u>Malgranges theorem</u> (M_1) if the kernel of Q is the closed linear span of $\{e^{\varphi} : \langle e^{\varphi}, Q\rangle = 0,\ \ \varphi \in E^*\}$ with respect to the topology τ_o and Q satisfies <u>Malgranges theorem</u> (M_2) if Q is onto.

Chapter 2

History and Motivation

2.1 Introduction

This chapter presents a scenario of developments in finite and infinite dimensional holomorphy and the theory of nuclear spaces; however, our treatment here is not exhaustive by all means. For the sake of simplicity we devide the whole chapter into three sections dealing respectively with finite dimensional holomorphy, infinite dimensional holomorphy and nuclear spaces.

2.2 Finite Dimensional Holomorphy

Before we pass on to digging out some historical aspects of this theory, it would be appropriate to mention here that finite dimensional holomorphy is nothing but the study of holomorphic mappings on finite dimensional spaces. No doubt, much is known about this theory and its origin dates back to Cauchy, but we mention here only the recent advances on the class of holomorphic mappings of several complex variables, related to their topological structure, series representation etc. We present the same in chronological order.

It is well known to analysts that the entire functions have two distinct representations in the form of series, namely (i) Taylor series and (ii) Dirichlet series. Indeed, to our knowledge it was Uryson [257] who considered the spaces of entire functions represented by Taylor's series in 1924. Later, in 1945, a systematic study of the spaces of analytic functions in a disc was made by Markusevic [165] who proved that the spaces of entire functions, endowed with the topology of uniform convergence on closed set of points of the disc, is a Frèchet space. A few years later, Iyer [122], [123], [124], [125], [126] took up the study of different spaces of entire functions, for example, the spaces of entire functions having finite growth properties etc. (cf. [126]) and studied various topological aspects of the spaces including the basis representation of entire functions relative to the topology generated by the metric defined by the coefficients occuring in the Taylor series expansion. This study was further continued by Arsove [8], [9] and Krishnamurty [156] who made systematic study for spaces of analytic functions on a disc.

On the other hand, a topological study of the spaces of entire functions represented by Dirichlet series was picked up by Kamthan [130], Husain and Kamthan [120] around 1968, who proved that such spaces are Montel.

Later in 1972, Gautam [94] made further investigations for these spaces related to the basis representation of their elements.

Almost at the same time, Kamthan [131] considered the space of entire functions of two variables, equipped with the topology generated by the paranorm $||f||$ defined as

$$||f|| = \sup \{|a_{oo}|; |a_{mn}|^{1/m+n}, m,n \geq 0, m+n \geq 1\}$$

where

$$f(z_1,z_2) = \sum_{m \geq 0} \sum_{n \geq 0} a_{mn} z_1^m z_2^n, \quad z_1, z_2 \in ¢ .$$

Besides proving the results on bases, he also showed that it is a Fréchet space. In 1973 Gupta [106] further carried out a similar study for entire functions having finite growth and the space of analytic functions on polydiscs; cf. also [135], [136], [137].

Around the year 1976, Kamthan [132] considered the class X of entire functions having order 1 and type 0, i.e. if $f \in X$, then

$$f(z) = \sum_{n \geq 0} a_n z^n, \quad |a_n|^{1/n} \to 0 \quad \text{as} \quad n \to \infty$$

and

$$\lim_{n \to \infty} [n! |a_n|]^{1/n} = 0.$$

After topologizing this space suitably, he proved its Frêchet character along with some results on basis.

Regarding the dual of the space $H(\mathbb{C}^n)$ of holomorphic mappings defined from $\mathbb{C}^n$ to $\mathbb{C}$, we have a characterization due to Ansemil contained in [3].

Recently in [134], a different approach to study the spaces of entire functions of two variables having finite order and type zero have been considered via the use of matrix spaces.

2.3 Infinite Dimensional Holomorphy

As the name suggests, infinite dimensional holomorphy is the study of holomorphic mappings defined on open, or compact, or more general subsets of complex locally convex spaces. Essentially, it was L. Nachbin [189] who developed the present theory of infinite dimensional holomorphy. In fact his aim was to extend the finite dimensional distribution theory of Schwartz [241], linear partial differential operators of Hormander [118] and Treves [255] and analytic functionals of Martineau [166]. This was largely accomplished by Nachbin [189] in 1963 through a series of lectures which envelop the basis on multilinear operators and fundamentals of analytic functions from Banach space to themselves. At the behest of the development of nuclear spaces

essentially due to Grothendieck [101] and the classical work of Malgrange [164] on convolution equations on the space of entire functions. Nachbin and Gupta [102] obtained, on a suggestion from the former to the latter, several basic results on the structure of a certain vector subspace $H_{Nb}(E,\mathbb{C})$, the space of nuclearly entire functions of bounded type (c.f. Chapter 6, Section 2, the space $H_N^{\mu}(E)$ for $\mu = c_o$ of the space $H(E,\mathbb{C})$ of entire functions on a complex Banach space to $\mathbb{C}$. For instance, let us recall the following from [102].

(i) The Borel transform sets a one to one correspondence between $(H_{Nb}(E,\mathbb{C}),\tau_o)^*$ and the vector subspace $\exp E^*$ of all entire function of exponential polynomial type.

(ii) For the convolution operator $Q:H_{Nb}(E,\mathbb{C}) \to H_{Nb}(E,\mathbb{C})$, every solution of a convolution equation $Q=O$ can be approximated by a finite sum of exponential polynomial solutions of the equation and $QH_{Nb}(E,\mathbb{C}) = H_{Nb}(E,\mathbb{C})$, i.e. Q is onto.

It was also found useful to distinguish between various types of mappings such as compact, nuclear, integral etc. Since holomorphic functions can be uniquely represented in terms of homogeneous polynomials satisfying certain conditions, one may define various subspaces of the space of analytic functions which

depend upon a particular class of polynomials, for instance one may refer to the work of Gupta [103], [104], [105]. Motivated by his disciple's contribution in these papers, around 1969, Nachbin [191] brought forth the concept of θ-holomorphy. This concept of θ-holomorphy is accountable to further researches in the theory of holomorphic functions on Banach spaces. Notably among the early contributors in these directions are Alexander [1], Aron [5], Barraso [12], Bochnak and Siciyak [25], [26], [27]; Boland [28]; Bremermann [45], [46]; Chae [47], Dineen [57], [59], [60]; Dwyer [85], Greenfield and Wallach [96], Matos [168], Meise [175], Rickart [222], [224], Schottenloher [239] and many others.

Later on, analysts studied infinite dimensional holomorphy in more or less four directions by way of (A) restricting the underlying spaces, (B) restricting the representative polynomial coefficients, (C) combining (A) and (B), and (D) studying various topologies on the space of holomorphic functions.

In what follows, we briefly touch upon the foregoing aspects (A) through (D).

(A) The development in this category is basically in two directions, namely, when E is a Banach space or a Hilbert space and secondly when E is an arbitrary locally convex space or a topological vector space, as indicated by the following diagram.

$$\begin{tikzcd} & H(E,F) \arrow[d] & \\ & H(E,\mathbb{C}) \arrow[dl] \arrow[d] \arrow[dr] & \\ H(B,\mathbb{C}) & H(H,\mathbb{C}) & H(E,\mathbb{C}) \end{tikzcd}$$

B = a Banach space; H = a Hilbert space;
E = arbitrary l.c. TVS or TVS and F = a Banach space

In what follows, we endeavour to present a chronological account of these two directions of development concerning (A).

In late sixty, Dineen [57] carried out rigorous investigations on different aspects of θ-holomorphy. In particular, he obtained conditions which force a pseudoconvex domain (i.e., a domain X in $\mathbb{C}^n$ on which a continuous function having its restrictions on discs in X as subharmonic which is bounded below) to become a domain of holomorphy and generalized the Cartan-Thullen theorem, namely, $X \subset \mathbb{C}^n$ is a region of holomorphy if and only if for each compact set $K \subset X$, the set $\{x \in X: |f(x)| \leq ||f||_K , \forall f \in H(X)\}$ is compact, from $\mathbb{C}^n$ to a Banach space. Later in [58], [59], [64] he studied several structures on the space H(U) where U is an open subset of an l.c. TVS or a Banach space E with Schauder base or decomposition, useful for the characterization of certain Runge domains which play an important

role in the study of analytic continuation in infinite dimension. For further researches in this direction one may refer to Dineen [57], [61]; Hirsowitz [116]; Nachbin [193]; and Noverraz [203], [204].

At the same time, attempts were made to strengthen some of the earlier results on $H(B,\mathbb{C})$, when the domain space B in $H(B,\mathbb{C})$, is replaced by a Hilbert space. Work in this direction was carried out by Harris [111] and Dwyer [85], [86].

During the years 1972-75, sufficient attention was also paid to discover result on convolution operators and Borel transform. In particular, let us recall the following theorems of Nachbin [193] and Boland [29] respectively:

''A complex valued function on B^* is the Borel transform of a continuous linear form on $H_{Nb}(B,\mathbb{C})$ if and only if it is entire function of exponential type on B^*''.

''Kernel of any convolution operator on $H_{Nb}(B,\mathbb{C})$, the weighted subspace of nuclearly entire functions of bounded type, is the closure of the subspace generated by the exponential polynomial functions in the Kernel''.

On the other hand, Dineen [64] took the initiative of extending the holomorphy theory on Banach spaces to

the setting of locally convex spaces, which was further carried out by Boland. To single out a few important results on $H(E,\mathbb{C})$, E an l.c. TVS, let us quote the following from [31] and [33] respectively.

''If E is a quasi-complete nuclear, or dual nuclear or dual of a Frechet-nuclear (i,e a DFN space) space, then the space $(H(E,\mathbb{C}),\tau_o)$ admits convolution operators satisfying Malgranges Theorems''.

''If E is a quasi-complete nuclear space, then $(E^*,\beta(E^*,E))$ is nuclear if and only if the space $(H(E,\mathbb{C}),\tau_o)$ is nuclear. Further, if E is a DFN-space and G a closed subspace of E, then the restriction map $H(E,\mathbb{C}) \to H(G,\mathbb{C})$ is a continuous, linear, open and onto map''.

Towards 1973 Nachbin [196] examined the factorization of holomorphic functions defined on the inductive limit of an increasing sequence of normed linear spaces $\{E_n\}$ such that every bounded subset in E_n is relatively compact in E_{n+1}, i.e. a Silva space.

Concerning analytic continuation and extensions of domains of holomorphy we would like to put on record the work of Aurich [10]; Dineen [60], [63], [66], [67]; Barraso, Matos and Nachbin [13], Herve [112], Mujica [187], Nachbin [195], [197]; Pincku [211] and Rickart [223], [224].

The years from 1975 to 1988 have been quite creative for the development of holomorphy in the setting of locally convex spaces, more specifically nuclear spaces; for instance, we may mention the contribution of Bierstedt and Meise [19], [20]; Boland [36], [37]; Boland and Dineen [39], [40], [41], [42]; Borgens Meise and Vogt [43], [44]; Dineen [69], [70], [72], [73], [74], Dwyer [86] and Mujica [186]. To have an insight into the recent advancement, let us quote a few important results of these mathematicians in the chronological order.

To begin with Boland [36] in 1976 proved the existence of a Schauder base in $(\mathcal{P}(^m E, \mathbb{C}), \varepsilon)$ where E is a Fréchet nuclear space with its dual E having a Schauder base, and ε is the topology of uniform convergence on bounded sets of E.

Around 1978, Boland and Dineen [40] exhibited the basis character of monomials [cf. Chapter 1] for the space $(H(\Lambda(P), \mathbb{C}), \tau_o$ or $\tau_\omega)$, where $\Lambda(P)$ is a fully nuclear Köthe space. They also proved the following:

''If U denote an open polydisc in the fully nuclear space $E = \Lambda(P)$ with a basis, then the strong dual of $(H(U), \tau_o)$ is algebraically isomorphic to the space $H(U^M)$ of holomorphic germs on the compact polydisc U^M; where $U^M = \{(\omega_n)_{n=1}^{\infty} \in E^*: \sup_n |\omega_n z_n| \leq 1, \forall\, z = (z_n)_{n=1}^{*} \in U\}$;

and the strong dual of $(H(U),\tau_\omega)$ is algebraically isomorphic to the space $H_{HY}(U^M)$''.

Almost three years later, Borgens , Meise and Vogt [43] contributed the following, regarding the study of holomorphy on power series spaces of infinite type.

''The spaces $(H((S^*,\beta(S^*,S)),\mathbb{C}),\tau_0)$ and $(H(H(\mathbb{C}^N,\mathbb{C}))^*_\beta,\tau_0)$ are topologically isomorphic to the spaces respectively S and $\Lambda_\infty((\ln(n+1))^{N+1/N})_{n\in\mathbb{N}}$, where S is the space of all rapidly decreasing sequences. Further $(H(\Lambda_\infty(\alpha))^*_\beta,\tau_\delta)$ is isomorphic to the space $\bigoplus \Lambda_\infty(\beta(\alpha))$ for any stable nuclear power series space $\Lambda_\infty(\alpha)$, $\beta(\alpha)$ being a suitably choosen sequence''.

In a subsequent paper [44], they also proved the $\Lambda(\beta(\alpha))$-nuclearity of $H(\Omega,\mathbb{C})$, Ω open in E, E being a quasi-complete l.c. TVS such that $(E^*,\beta(E^*,E))$ is $\Lambda(\alpha)$-nuclear where $\Lambda=\Lambda_\infty$, Λ_1 or Λ_N and also $\Lambda(\beta(\alpha))$-nuclearity of the strong dual of the space $H(K)$ of holomorphic germs under suitable restriction.

(B) As mentioned earlier, the nature of coefficient polynomials play an important role in enriching the structure of various subspaces of holomorphic mappings. To **unify** the study of various subspaces, Nachbin [191] introduced the notion of θ-holomorphy (cf. Chapter 1). However, concerning this aspect of study in holomorphy,

it will be worthwhile to recall a few basic results of Dineen obtained in [57]. In fact, he introduced the concept of α-holomorphy type, α-β-holomorphy type and α-β-γ-holomorphy type. For α-holomorphy, he showed that the topology $\tau_{\omega\alpha}$ arising out of α-holomorphy is the finest locally convex topology which induces on each subspaces of homogeneous polynomial its original norm topology and for which the Taylor series converges absolutely. For α-β-holomorphy type, he characterized the dual spaces using the Borel transforms. Further for α-β-γ-holomorphy type, he proved that the partial differential operators are onto maps and the solutions can be approximated by exponential polynomial solutions.

Almost at the same time, another promising direction was added by Chae [47] who introduced the space $H_{\Theta}(K,F)$ of germs of holomorphic mappings around a compact set K for a holomorphy type θ

Around 1973-74, the investigations of Boland [29], [30] and Dwyer [85], [86] regarding the study of convolution equation on the space $H_{Nb}(E^*,F)$, gave rise to the development of Hilbert-Schmidt type and nuclear type holomorphy. However, for holomorphy types on open subsets of Banach spaces, we may refer to the work of Aron [5].

Another significant contribution was from Nachbin and Dineen [199], who basically proved that a complex valued function on E^* is the Borel transform of a continuous linear form on $H_{Nb}(E,\mathbb{C})$. They also answered an open problem posed by Nachbin himself in [193], by giving an example of an entire function of exponential type on E^*, which is not the Borel-transform of any continuous linear form or $H_{NC}^{c_o}(E)$, the space of holomorphic mapping f with nuclear compact polynomials $\hat{d}^n f(0)$ such that $\{(\frac{\|\hat{d}^n f(0)\|_N}{n!})^{1/n}\} \in c_o$.

Whereas a good account of compact polynomial and compact holomorphic maps between Banach spaces is to be found in the work of Aron [6]; the composition of polynomials is dealt with in the work of Stevenson [247].

Around 1978, Colombeau, Meise and Perrot [62] introduced the concept of Silva holomorphic maps (i.e. the mapping defined by the formal power series of polynomials in $\mathcal{P}_a(^mE)$ (cf. Chapter 1) and proved that, . holomorphic functions are dense in the space of Silva holomorphic functions with respect to the compact open topology''. Zaine [266] also investigated the envelop of holomorphy, at the same time.

A remarkable contribution in this direction, appeared in [51], is due to Colombeau and Matos, who

observed that the space $H_{Nb}(E)$ of nuclearly entire functions of bounded type and $H_{sNb}(E)$ of nuclear Silva entire functions of bounded type are algebraically and topologically isomorphic to the space $H_s(E)$ of Silva entire functions. This study on the space of nuclear Silva entire functions and [172] encouraged Matos, Nachbin [173] to carry out the researches of holomorphy type to the setting of locally convex space and there by introducing the theory of Silva holomorphy type.

In [18] , using the study of Silva holomorphic functions on complex l.c. TVS, Bianchini generalized the results of Dineen [57].

Finally, we quote the work of Ansemil and Ponte [4], who gave an example of a quasi-normable Fréchet space which is not Schwartz. Indeed, they proved ,

''$(H_b(U,F)\ \tau_b)$, the space of holomorphic mappings of bounded type, is a Montel, Schwartz, or a nuclear space if and only if E and F are finite dimensional, where τ_b is the natural tpology arising out of boundedness''.

(C) This part is a continuation of (A) and (B). We have already come across with most of the contributions made in this direction either in (A) or (B). However,

just to make the list more exhaustive, we mention a few more in chronological order.

Around 1960, Bremermann [45] studied domains of holomorphy. In late sixties and early seventies Aron and Cima [7] ; Bochnak [23] ; Bochnak and Siciak [25] ; Boland and Dineen [38] ; Coeure [49]; Colombeau [50]; Dineen [58], [60], [65] ; Dineen and Nachbin [76]; Gupta [105]; Hirshowicz [114], [115]; Lelong [161]; Matos [169], [170], [171], Nachbin [190], [192], [193], [194], [195]; Noverraz [202], [203]; Rickart [223]; Siciak [243] contributed to various aspects of holomorphy, for instance, Dineen [64] proved that

''$(H(U), \tau_\omega)$ is complete if and only if the space $(\mathcal{P}(^mE), \tau_\omega)$ is complete for each m and if $P_m \in (\mathcal{P}(^mE), \tau_\omega)$ such that $\{p(P_m)\} \in \ell^1$ for each τ_ω-continuous seminorm p on H(U), then $\sum_{m\geq 0} P_m \in H(U)$''.

From the year 1975 onwards, analysts working on holomorphic mappings got interested to pursue the investigations of holomorphic mappings defined on nuclear or dual nuclear spaces. In this direction we may refer to Bierstedt and Meise [19], Boland [33], [34], [37]; Boland and Dineen [38] , [41]; Colombeau and Perrot [55]; Dineen [69], [70], [74]. Dineen, Meise and Vogt [75]; Dinnen and Nachbin [76] ; Matos and Nachbin [172] ;

Meise and Vogt [177] ; Waelbroeck [261]. We quote from [40] the following

''If E is fully nuclear space with a basis then strong dual of $(H(U), \tau_o$ or $\tau_\omega)$ is isomorphic to the space of holomorphic **germs** on compact poly**disc** $U^M \subset E^*$''.

Another direction of researches on holomorphic mappings defined on Banach, locally convex and metrizable spaces centres around the work of Colombeau, Meise, Mujica and Vogt. Indeed Colombeau and Mujica [53] proved that, there always exists an analytic function which can be approximated by a formal power series.

Lastly, it will be worthmentioning the contributions of Barraso and Nachbin [15]; Bierstedt and Meise [20] ; Boland [30]; Bremermann [46]; Coeure [48]; Colombeau and Matos [51] , Dineen [73] , [74] ; Dineen, Meise and Vogt [75] ; Lelong [160] ; Matyszczyk [174] ; Meise [176]; Nachbin [198], Rolewicz [229] and Silvano and Trisi [244].

(D) Regarding the topological aspects of the class of holomorphic functions, let us go back to the days of Laurent Schwartz [241] who introduced the concept of inductive limit topology on the space H(E) of analytic functions. Later in [190], Nachbin observed that the compact open topology and the inductive limit topology

on this space were insufficient to derive certain properties for instance, completeness of $H(U)$ on certain domains of holomorphy and holomorphic extensions. This led him to introduce the ported topology τ_ω and countable compact open topology τ_δ which were thoroughly investigated by Dineen [64], for the general case of holomorphy types. Dineen [67], [68] also considered the Chae's projective limit topology and proved results relating to these topologies. Some of his deep results regarding the topological structure of the space $H(U)$ from [64] are quoted as

''On the space $H(U)$, $(H(U), \tau_{\omega,b}) = (H(U), \tau_\delta) \Longleftrightarrow (H(U), \tau_{\omega,b})$ is barrelled $\Longleftrightarrow$ $\tau_{\omega,b}$ is the finest locally convex topology on $H(U)$ for which the Taylor's series converges absolutely and which induces on $\mathcal{P}(^mE)$ the τ_ω-topology for each m''.

''$(H(U), \tau_{0,b})$ is barrelled if and only if $(\mathcal{P}(^mE), \tau_{0,b})$ is barrelled for each m and $\tau_{0,b}$ is the finest locally convex topology on $H(U)$ for which the Taylor's series converges absolutely and which induces on $\mathcal{P}(^mE)$ the $\tau_{0,b}$ topology for each m''.

''$(H(E), \tau_{\omega,b})$ is a Montel space whenever E is Fréchet nuclear''.

In addition to these results he also investigated conditions under which the topologies τ_0, τ_ω and τ_δ

are equivalent and the space $H(U)$ equipped with τ_o or τ_ω or τ_δ becomes bornological, Schwartz and nuclear.

Around 1974 Nachbin [197], besides giving some relations between the projective and inductive limit topologies, also studied the impact of various topologies on holomorphy. He, in collaboration with Barraso [15] also characterized the bounded sets of $H(U,F)$ with respect to the topologies τ_o or τ_ω or τ_δ.

Almost three years later, Boland [36] proved the equivalence between the nuclearity of $(E^*,\beta(E^*,E))$ and that of $(H(E),\tau)$ for $\tau = \tau_o$, τ_ω or τ_δ, for a quasi-complete dual nuclear space E. He also posed a problem regarding Hahn-Banach type extension theorem for the space of entire functions with respect to the topologies τ_ω and τ_δ.

In case of an open subset U of $\mathbb{C}^I$, Barraso and Nachbin [14] proved

''I is countable (resp. finite) if and only if $\tau_\omega = \tau_o$ (resp. $\tau_\omega = \tau_{\omega,b}$), for the space $H(\mathbb{C}^I,\mathbb{C})$''.

Boland and Dineen [40] extended the above result namely, the equivalence between τ_o and τ_ω, to the space of entire complex valued function defined on the dual of a Frèchet nuclear space. Concerning the basis representation of members in $H(E)$, they proved that the

space H(E) equipped with τ_o or τ_ω possesses a Schauder basis for a fully nuclear Köthe space E and further obtained the Schwartz, Montel and nuclear character of the space H(U) relative to τ_o and τ_ω.

Using a result of Dineen [64] that H(U) possessess a τ_ω-Schauder or τ_δ-Schauder decomposition, Isidro [121] established many topological properties for this space for instance, he proved

"If $(H(U), \tau_\omega)$ is complete, then so is the space $(H(U), \tau_\delta)$".

By using the idea of inductive limit topology, Mujica [186] simplified the work of Chae's who proved results on holomorphic germs with respect to the topology τ_ω.

Almost at the same time, Bierstedt and Meise [20] investigated that H(K) has the inductive limit topology of the system $(H(U), \tau_o)_{U \subset K}$ and obtained several results concerning nuclearity and Schwartz property of H(K).

Using the holomorphic germs technique and inductive and projective limit topologies Boland and Dineen further generalized the work of [68] in [39].

A good account of topological aspects of space of holomorphic mappings on Banach spaces as well as l.c. TVS, has been provided by Dineen in his book [73]. He related

the topology τ_δ with τ_ω and τ_o in the following form

''If E is a Banach space with an unconditional basis then $\tau_\delta = \tau_\omega$ on H(U), for an open and balanced subset U of E''.

''If a nuclear space E is isomorphic to the space S of rapidly decreasing sequences, then $\tau_\delta = \tau_o$ on H(E)''.

Almost at the same time, Katsaras [143] proved that

''τ_o τ_ω and τ_δ are equivalent on C(E,X), space of continuous functions from E to a locally convex space X where E is locally compact and completely regular (resp. E is a Schwartz space and X is complete)''.

On the other hand, Boland and Dineen [42] studied the relationship of the topologies τ_o, τ_ω and δ_δ on the space of holomorphic mappings defined on the spaces of tests functions and distributions. Indeed, they established.

''Let Ω be an open subset of $\mathbb{R}^n$ and $D(\Omega)$ be the Schwartz space of tests functions and $D^*(\Omega)$ be Schwartz space of distribution then $\tau_o = \tau_\omega = \tau_\delta$ on $H(D(\Omega))$ and $H(D(\Omega)) \underset{+}{\subset} H_{HY}(D(\Omega))$ but $H_{HY}(D^*(\Omega)) = H(D^*(\Omega))$ and $\tau_o \neq \tau_\omega \neq \tau_\delta$ on $H(D^*(\Omega))$''.

Recently, Jose [128] has obtained results similar to that of Katsaras for the space of bounded real valued continuous functions on completely regular spaces.

Besides we would also like to put on record the work of Boland [32]; Colombeau, Meise and Perrot [52]; Meise [176]; Meise and Vogt [178] and Moraes [183] made in this direction.

Finally a remark of Nachbin in [198] reveals that; the importance of holomorphy lies in its applications to Mathematical Physics, Electrical Engineering for studying Volterra operators, Mechanics and Economics; however, the ties between the existing theory and its applications are still loose because of the lack of interaction between mathematicians and users. We sincerely hope that the existing gap would be bridged as the wheel of time goes by.

2.4 Nuclear Spaces

The concept of nuclear spaces is an outcome of the notion of nuclear operators which have their origin in the work of Schatten [236] and Schatten and Von Neumann [237] who studied the same on separable Hilbert spaces as ''finite trace-class operators''. Indeed, it was Grothendieck [101] who, motivated by the famous kernel theorem of Schwartz [242], which characterizes a

bilinear functional on the space of infinitely differentiable function having bounded supports, introduced the notion of nuclear operators and nuclear spaces around the year 1953. He used the cumbersome technique of topological tensor products of l.c. TVS initiated by himself in his thesis, which was subsequently simplified by Pietsch [209] with the help of theories of vector valued sequence spaces and summability of families.

The theory of nuclear spaces attained further significance especially after Mityagin's work [179], [180] came to limelight. Indeed, inspired by a problem of Gelfand, he made use of the notions of n-th diameter [148] and ε-entropy [149] of Kolmogorov, diemetral dimension of Bessaga, Pelczynski and Rolewicz [17] and the approximative dimension of Kolmogorov [149] and Pelczynski [207], to give several characterizations of nuclear spaces.

Further contributions along these lines have also been made by Schock [238] and Fenske and Schock [88]. We also mention the important work of Terzioğlu [251], where the problem of diematral dimension has been subjected to further investigations and subsequently utilized in exploring the conditions for the nuclearity and Schwartz property of Köthe sequence spaces. For further details on Schwartz spaces and historical remark on this subject matter, we refer to [133].

The first universality theorem on nuclear spaces was proved by T. and Y. Komura [151] in 1965. They proved that every nuclear space can be embedded as a subspace of a suitable topological product of the nuclear space S of rapidly decreasing sequence, which substantiated a conjecture of Grothendieck, namely, ''Every nuclear space can be embedded in the product of power series spaces''. In this direction we would like to put on record the contributions of Saxon [234] and Fehr and Jarchow [87].

The book by Wong [265], which embodies, apart from the theory of nuclear spaces, the theory of Schwartz spaces and tensor products, is also a useful addition to the stock of literature on nuclear spaces. The monograph by Dubinsky [79] dealing with structure of nuclear Frechet spaces, is also worthy of special mention.

On the other hand, from application point of view of nuclearity, one is referred to [75], [147], [155], [157], [158], [205], [206], [259]. Indeed, the importance and significance of nuclear spaces lie mainly in the fact that, all infinite dimensional locally convex spaces with a few exceptions, encountered in analysis are either normed spaces or nuclear spaces; for instance,

they include the space of analytic, entire, harmonic, infinitely differentiable functions and the space of distributions of L. Schwartz, as particular cases; cf. [95], [209]. Further, the close association of nuclear spaces with holomorphy and a remark of Pietsch ''if a theory of structure for locally convex spaces can be developed at all, then it must certainly be possible for nuclear spaces because they are more closely related to finite dimensional spaces than the normed spaces'' , support our assertion.

Chapter 3

Köthe Matrix Space

3.1 Introduction

In this chapter we consider an extended class of analytic functions of several complex variables and investigate several topological properties of the same including the Schwartz property. For the sake of brevity we confine our attention to the two variables case only and attempt to extend the class of analytic functions of two variables in terms of infinite matrices. This new class $\Lambda(P)$ of infinite matrices envelops the space of analytic functions in the bi-cylinder considered in [135] as well as the space of entire functions of two variables initiated in [131]. The last two sections of this chapter are respectively devoted to proving the Schwartz property of the space $\Lambda(P)$ and an arbitrary locally convex space X. Where as in Section 3, we find the estimates of Kolmogoroff diameters for proving the Schock-Terzioglu criterion for the Schwartz property of $\Lambda(P)$; in Section 4 we prove the Schwartz property of an arbitrary locally convex space via a notion called diametrical dimension which is defined with the help of Kolmogoroff diameters.

3.2 Structural Properties

As mentioned above, we study in this section the class of analytic functions of two variables in the form of the space $\Lambda(P)$. Recalling the definition of Ω from Chapter I, we introduce

Definition 3.2.1: A set $P \subset \Omega$ is called a K-box if the following three conditions hold:

(i) For each x in P, $x_{mn} \geq 0$, $\forall\, m,n \geq 0$;

(ii) For x, y in P, there exists $z \in P$ such that $x_{mn} \leq z_{mn}$, $y_{mn} \leq z_{mn}$, $\forall\, m,n \geq 0$;

(iii) For each $(m,n) \in \mathbb{N}_0 \times \mathbb{N}_0$, there exists x in P with $x_{mn} > 0$.

Corresponding to a K-box P, the space $\Lambda(P)$ defined by

$$\Lambda(P) = \{x \in \Omega:\ p_a(x) = \sum\sum_{m+n \geq 0} |x_{mn}|\, a_{mn} < \infty,\ \forall\, a \in P\},$$

is called a Köthe matrix space. The family $\{p_a : a \in P\}$ generates a Hausdorff locally convex topology on $\Lambda(P)$ which we denote by T_P.

As a particular case of $\Lambda(P)$, we confine our attention to a specified type of Köthe box P which ultimately yields the space of entire functions of two variables (cf. Corollary 3.2.6). Indeed, choose α, β in ω satisfying

$$(3.2.2) \qquad 0 = \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \to \infty;\ 0 = \beta_0 \leq \beta_1 \leq \cdots \leq \beta_n \to \infty .$$

Define

$$P = P_{\alpha,\beta} = \{R_1^{\alpha_m} R_2^{\beta_n} : 0 < R_i < \infty \ , \ i = 1,2\}.$$

Then we denote the space $\Lambda(P_{\alpha\beta})$ by $\Lambda(\alpha,\beta)$.

Also, we introduce

Definition 3.2.3: For α,β in ω satisfying (3.2.2), the space $\delta^{\alpha,\beta}$ defined as follows

$$\delta^{\alpha,\beta} = \{x \in \omega : \lim_{m+n\to\infty} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0\}$$

is referred to as the space of generalized entire functions.

The relationship between $\Lambda(\alpha,\beta)$ and $\delta^{\alpha,\beta}$ is contained in

Proposition 3.2.4: Let $\alpha,\beta \in \omega$ be allowed to satisfy the E-condition, namely,

$$(3.2.5) \qquad \sum\sum_{m+n\geq 0} R_1^{-\alpha_m} R_2^{-\beta_n} < \infty \ ,$$

for some $R_1, R_2 > 1$. Then the Kothe matrix space $\Lambda(\alpha,\beta)$ concides with the space $\delta^{\alpha,\beta}$.

Proof: For proving $\Lambda(\alpha,\beta) \subset \delta^{\alpha,\beta}$, consider $x \in \Lambda(\alpha,\beta)$ and suppose

$$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \not\to 0 \quad \text{as} \quad m+n \to \infty$$

Hence for some $\varepsilon > 0$, there exist increasing sequences of

integers $\{m_i\}$ and $\{n_j\}$ such that

$$|x_{m_i n_j}| > \varepsilon^{\alpha_{m_i} + \beta_{n_j}}, \quad i+j \geq 0.$$

Choose R_1, R_2 so that $\varepsilon R_1, \varepsilon R_2 > 1$. Then we have

$$\sum\sum_{i+j \geq 0} |x_{m_i n_j}| \, R_1^{\alpha_{m_i}} R_2^{\beta_{n_j}} \geq \sum\sum_{i+j \geq 0} (\varepsilon R_1)^{\alpha_{m_i}} (\varepsilon R_2)^{\beta_{n_j}} = \infty ,$$

which is a contradiction to the fact that $x \in \Lambda(\alpha,\beta)$. Thus $\Lambda(\alpha,\beta) \subset \delta^{\alpha,\beta}$.

For the other inclusion, let $x \in \delta^{\alpha\beta}$. Then

$$(+) \qquad \lim_{m+n \to \infty} |x_{mn}|^{\frac{1}{\alpha_m + \beta_n}} = 0.$$

The condition (3.2.5) is equivalent to the following,

$$\sum\sum_{m+n \geq 0} u_1^{\alpha_m} u_2^{\beta_n} < \infty ,$$

for some u_1, u_2 with $0 < u_i < 1$, $i=1,2$. Then for any R_1, R_2 with $0 < R_i < \infty$, choose $\varepsilon > 0$ so that

$$\varepsilon R_1 < u_1, \ \varepsilon R_2 < u_2$$

By (+) there exists integer N such that

$$|x_{mn}| < \varepsilon^{\alpha_m + \beta_n}, \quad m+n > N.$$

Hence, we have

$$\sum\sum_{m+n>N} |x_{mn}| R_1^{\alpha_m} R_2^{\beta_n} \leq \sum\sum_{m+n>N} (\varepsilon R_1)^{\alpha_m} (\varepsilon R_2)^{\beta_n}$$

$$\leq \sum\sum_{m+n>N} u_1^{\alpha_m} u_2^{\beta_n}$$

$$\leq \sum\sum_{m+n\geq 0} u_1^{\alpha_m} u_2^{\beta_n}$$

$$< \infty$$

Therefore, $x \in \Lambda(\alpha,\beta)$; that is, $\delta^{\alpha,\beta} = \Lambda(\alpha,\beta)$.

Corollary 3.2.6: For $\alpha_m = m$, $\beta_n = n$, $\Lambda(\alpha,\beta) = \delta^{\alpha,\beta}$ is the space δ of all entire functions of two complex variables considered in [131].

Observe that the condition (3.2.5) holds for all $R_1, R_2 > 1$.

The Topology on $\delta^{\alpha,\beta}$:

For $x \in \delta^{\alpha,\beta}$, define

(3.2.7) $$||x||_{\alpha,\beta} = \sup \{ |x_{00}|;\ |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}},\ \forall\, m,n \geq 1 \}.$$

Then we have

Proposition 3.2.8: $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is an F-normed space where $T_{\alpha,\beta}$ is the topology defined by $||\cdot||_{\alpha,\beta}$.

Proof: It is sufficient to prove that $||\cdot||_{\alpha,\beta}$ defines an F-norm on $\delta^{\alpha,\beta}$. For $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$, the inequality $||\lambda x||_{\alpha,\beta} \leq ||x||_{\alpha,\beta}$ follows trivially from the definition (3.2.7).

The triangle inequality, namely,

$$||x+y||_{\alpha,\beta} \leq ||x||_{\alpha,\beta} + ||y||_{\alpha,\beta}$$

is a consequence of the inequality

$$|x_{mn}+y_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \leq |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} + |y_{mn}|^{\frac{1}{\alpha_m+\beta_n}}$$

(cf. [152] p. 158) which is true for all $m,n \geq 1$.

Next, we show that $||\lambda x^p||_{\alpha,\beta} \to 0$ whenever $||x^p||_{\alpha,\beta} \to 0$ and $\lambda \in \mathbb{K}$. As the double sequence, $\{|\lambda|^{\frac{1}{\alpha_m+\beta_n}}\}$ is convergent, we can find $M > 0$ such that

$$|\lambda|, \quad |\lambda|^{\frac{1}{\alpha_m+\beta_n}} \leq M, \ \forall\, m,n \geq 1.$$

Now for given $\varepsilon > 0$, there exists $p_o \equiv p_o(\varepsilon)$ such that

$$||x^p||_{\alpha,\beta} < \frac{\varepsilon}{M}, \ p \geq p_o.$$

Therefore

$$||\lambda x^p||_{\alpha,\beta} = \sup\{|\lambda x^p_{oo}|;\ |\lambda x^p_{mn}|^{\frac{1}{\alpha_m+\beta_n}},\ m+n \geq 1\}$$
$$< \varepsilon$$

for all $p \geq p_o$.

Finally, we need prove that $||\lambda_i x||_{\alpha,\beta} \to 0$, whenever $\{\lambda_i\} \in \mathbb{K}$ with $\lambda_i \to 0$ and $x \in \delta^{\alpha,\beta}$. So consider x in $\delta^{\alpha,\beta}$ and choose $\varepsilon > 0$. Then there exists an integer

$N_o \equiv N_o(\varepsilon)$ such that

(+) $$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon, \; m+n > N_o$$

Let

$$M = \max \{|x_{oo}|; \; |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, \; 1 \le m+n \le N_o\}$$

As $\{\lambda_i\}$ is a null sequence, we can find an integer i_1 such that

(*) $$|\lambda_i| < \min \{1, \tfrac{\varepsilon}{M}\}, \; \forall \, i \ge i_1 .$$

Also for fixed m and n we see that

$$|\lambda_i|^{\frac{1}{\alpha_m+\beta_n}} \to 0 \quad \text{as} \quad i \to \infty .$$

Hence for all m,n in $\mathbb{N}_o$ satisfying $1 \le m+n \le N_o$, there exists an integer $i_2 \equiv i_2(\varepsilon)$ such that

(++) $$|\lambda_i|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon/M, \quad \forall \, i \ge i_2 .$$

Let

$$i_o = \max \{i_1, i_2\}.$$

Then for $i \ge i_o$,

$$|\lambda_i x_{oo}|, \; |\lambda_i x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon, \; m+n > N_o$$

from (+) and (*), and

$$|\lambda_i x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon, \; 1 \leq m+n \leq N_o$$

from (++). Consequently

$$||\lambda_i x||_{\alpha\,\beta} < \varepsilon, \; \forall\, i \geq i_o .$$

This completes the proof.

In view of Propositions 3.2.4 and 3.2.8 we have two topologies T_p and $T_{\alpha,\beta}$ on the space $\delta^{\alpha,\beta}$. For the comparison of these two topologies, we make use of Mandelbrojt restriction on α as well as on β, namely,

$$\text{(3.2.9)} \qquad \underline{\lim}_{m\to\infty} (\alpha_m-\alpha_{m-1}) = 2h; \; \underline{\lim}_{n\to\infty} (\beta_n-\beta_{n-1}) = 2k,$$

where $h,k > 0$ are finite numbers.

In this section we shall assume throughout that α,β satisfy the condition (3.2.9). Then we have

Proposition 3.2.10: The topologies T_P and $T_{\alpha,\beta}$ on $\delta^{\alpha,\beta}$ are equivalent.

Proof: For showing $T_P \subset T_{\alpha,\beta}$ consider a sequence $\{x^p\}$ in $\delta^{\alpha,\beta}$ such that $x^p \to x$ in $T_{\alpha,\beta}$. Take $R_1, R_2 > 0$ and $\varepsilon > 0$ arbitrarily. One can find an integer N such that

$$\alpha_n-\alpha_{n-1} \geq h; \; \beta_n-\beta_{n-1} \geq k, \; \forall\, n \geq N.$$

Choose $\eta > 0$ so small that ηR_1, $\eta R_2 < 1$ and

$$\eta+ \frac{(\eta R_1)^{\alpha_1}(\eta R_2)^{\beta_1}}{[1-(\eta R_1)^h][1-(\eta R_2)^k]}[N(1-(\eta R_1)^h)+(\eta R_1)^h][N(1-(\eta R_2)^k)+(\eta R_2)^k]<\varepsilon$$

Now for $p \geq Q = Q(\eta)$,

$$|x^p_{oo}-x_{oo}| < \eta; \quad |x^p_{mn}-x_{mn}| < \eta^{\alpha_m+\beta_n}, \quad \forall\, m,n \geq 0.$$

Hence for $p \geq Q$, we have

$$p_{R_1,R_2}(x^p-x) = \sum\sum_{m+n\geq 0} |x^p_{mn}-x_{mn}| R_1^{\alpha_m} R_2^{\beta_n}$$

$$= |x^p_{oo}-x_{oo}| + \sum\sum_{m+n\geq 1} |x^p_{mn}-x_{mn}| R_1^{\alpha_m} R_2^{\beta_n}$$

$$< \eta + \sum\sum_{m+n\geq 1} (\eta R_1)^{\alpha_m}(\eta R_2)^{\beta_n}$$

$$\leq \eta+\Big(\sum_{i=1}^{N-1}(\eta R_1)^{\alpha_i}+ \frac{(\eta R_1)^{\alpha_N}}{1-(\eta R_1)^h}\Big)\Big(\sum_{j=1}^{N-1}(\eta R_2)^{\beta_j} + \frac{(\eta R_2)^{\beta_N}}{1-(\eta R_2)^k}\Big)$$

$$\leq \eta+ \frac{(\eta R_1)^{\alpha_1}(\eta R_2)^{\beta_1}}{[1-(\eta R_1)^h][1-(\eta R_2)^k]}(N(1-(\eta R_1)^h)+(\eta R_1)^h)(N(1-(\eta R_2)^k)+(\eta R_2)^k)$$

$< \varepsilon$

Thus $T_p \subset T_{\alpha,\beta}$.

For the converse, suppose now that $x^p \to x$ in T_p. Then in particular, $x^p_{oo} \to x_{oo}$. Consider $\varepsilon > 0$ and choose R_1,R_2 such that $R_1,R_2 \geq \frac{1}{\varepsilon}$, then there exists $Q=Q(\varepsilon,R_1,R_2)$ such that

$$|x_{00}^p - x_{00}| < \varepsilon \ ; \ p_{R_1,R_2}(x^p - x) < 1,$$

for $p \geq Q$. Hence for all m,n with $m+n \geq 1$ and $p \geq Q$

$$|x_{mn}^p - x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \leq \frac{1}{R_1^{\frac{\alpha_m}{\alpha_m+\beta_n}} R_2^{\frac{\beta_n}{\alpha_m+\beta_n}}} \leq \frac{1}{R} \leq \varepsilon,$$

where $R = \min\{R_1,R_2\} \geq \varepsilon^{-1}$, giving $x^p \to x$ in $T_{\alpha,\beta}$. Hence $T_{\alpha,\beta} \subset T_P$. Therefore $T_{\alpha,\beta} \approx T_P$.

<u>Proposition</u> 3.2.11: The space $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is complete.

<u>Proof</u>: It is sufficient to show that for each Cauchy sequence $\{x^p\}$ in $\delta^{\alpha,\beta}$, there corresponds a unique $x \in \delta^{\alpha,\beta}$ such that $||x^p - x|| \to 0$. For $\varepsilon > 0$, we find $Q = Q(\varepsilon)$ such that

$$||x^p - x^q|| < \varepsilon, \ \forall \ p,q \geq Q$$

In other words

$$(*) \qquad |x_{00}^p - x_{00}^q|, \ |x_{mn}^p - x_{mn}^q|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon, \ \forall \ p,q \geq Q; \ m+n \geq 1;$$

and so $\{x_{mn}^p\}$ is Cauchy in $\mathbb{K}$ for all $m,n \geq 0$. Since $\mathbb{K}$ is complete, let $x_{mn}^p \to x_{mn}$, say, as $p \to \infty$ for each pair $m,n \geq 0$. But we have

$$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \leq |x_{mn}^p - x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} + |x_{mn}^p|^{\frac{1}{\alpha_m+\beta_n}}$$

Hence fixing p(say $p=Q$) in the above inequality and using the fact that

$$|x_{mn}^{p}|^{\frac{1}{\alpha_m+\beta_n}} \to 0 \quad \text{as} \quad m+n \to \infty,$$

we get

$$\lim_{m+n\to\infty} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0$$

Thus $x \in \delta^{\alpha,\beta}$. Using (*) once again, we see that

$$||x^p - x||_{\alpha,\beta} \leq \varepsilon, \; p \geq Q.$$

<u>Lemma</u> 3.2.12: The space $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is non-normable.

<u>Proof</u>: For proving the result, we need prove that no neighbourhood of 0 is $T_{\alpha,\beta}$ bounded. Therefore consider an arbitrary 0-neighbourhood G in $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$. Then for some $\varepsilon > 0$, we get

$$\{x : p_{R_1,R_2}(x) < \varepsilon\} \subset G.$$

Define $x^p \in \delta^{\alpha,\beta}$ by

$$x^p = \frac{\varepsilon}{2}\left(\frac{1}{R_1}\right)^{\alpha_p}\left(\frac{1}{R_2}\right)^{\beta_p} e^{pp}.$$

Then $x^p \in G$ for $p \geq 1$. If $\varepsilon_p = 2^{-(\alpha_p+\beta_p)}$, then

$$p_{2R_1,2R_2}(\varepsilon_p x^p) = \varepsilon/2 > \varepsilon/4,$$

and so

$$\varepsilon_p x^p \notin \{x \in \delta^{\alpha,\beta} : p_{2R_1,2R_2}(x) < \varepsilon/4\}$$

leading to the fact that $\varepsilon_p x^p \not\to 0$. Hence G is not bounded with respect to T_p and consequently with respect to $T_{\alpha,\beta}$ by Proposition 3.2.10.

We are now prepared to state the main result in the form of

Theorem 3.2.13: $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is a non-normable Frêchet space.

Proof: This follows from Propositions 3.2.10 and 3.2.11 and Lemma 3.2.12.

Remark 3.2.14: In particular, Theorem 3.2.13 includes the Theorem 2.1 of [131].

3.3 Estimation of Kolmogorov's Diameters

In order to characterize Schwartz character of an arbitrary Köthe metrix space $\Lambda(P)$, we first find in this section the estimates of Kolmogorov's diameters for the neighbourhoods at origin. To be precise in our discussion, for $u, v \in P$ with $u_{mn} \leq v_{mn}$, $m, n \geq 0$, let us denote by $\{\alpha_{mn}\}$ a member of Ω defined by,

$$(3.3.1) \qquad \alpha_{mn} = \begin{cases} \dfrac{u_{mn}}{v_{mn}}, & \text{when } v_{mn} \neq 0 ; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following estimation.

Proposition 3.3.2: Let $u_{mn} > 0$ for all m, n, then we have

$$\inf_{m+n\leq s} \alpha_{mn} \leq \delta_{\frac{s(s+3)}{2}}(U_v, U_u) \leq \delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u)$$

$$\leq \delta_{\frac{s(s+1)}{2}}(U_v, U_u) \leq \sup_{m+n\geq s} \alpha_{mn}$$

for $s \in \mathbb{N}$, where U_u and U_v are the closed unit balls relative to p_u and p_v respectively.

Proof: For fixed $s \in \mathbb{N}$, let us construct a space

$$L_s = \mathrm{sp}\,\{e^{mn} : 0 \leq m+n \leq s-1\},$$

If x is in U_v, we have

$$\begin{aligned} p_u(x-x^{(s-1)}) &= \sum_{m+n\geq s}\sum |x_{mn}|u_{mn} \\ &= \sum_{m+n\geq s}\sum |x_{mn}| \frac{u_{mn}}{v_{mn}} \cdot v_{mn} \\ &\leq \sup_{m+n\geq s} \alpha_{mn} \sum_{m+n\geq s}\sum |x_{mn}|v_{mn} \\ &\leq \sup_{m+n\geq s} \alpha_{mn} \sum_{m+n\geq 0}\sum |x_{mn}|v_{mn} \\ &= \sup_{m+n\geq s} \alpha_{mn}\, p_v(x) \\ &\leq \sup_{m+n\geq s} \alpha_{mn} . \end{aligned}$$

Hence

$$x-x^{(s-1)} \in (\sup_{m+n\geq s} \alpha_{mn})\, U_u$$

$$\Longrightarrow \quad x \in L_s + (\sup_{m+n\geq s} \alpha_{mn})\, U_u$$

$$(*) \qquad \Longrightarrow \quad \delta_{\frac{s(s+1)}{2}}(U_v, U_u) \leq \sup_{m+n\geq s} \alpha_{mn} .$$

Next, let $x \in (\inf_{m+n \leq s} \alpha_{mn}) [U_u \sqcap L_{s+1}]$. Then for $x \in L_{s+1}$, we get

$$\begin{aligned}
p_v(x) &= \sum_{m+n \geq 0}\sum |x_{mn}| v_{mn} \\
&= \sum_{0 \leq m+n \leq s}\sum |x_{mn}| v_{mn} \\
&= \sum_{0 \leq m+n \leq s}\sum |x_{mn}| u_{mn} \alpha_{mn}^{-1} \\
&\leq (\sup_{m+n \leq s} \alpha_{mn}^{-1}) \sum_{0 \leq m+n \leq s}\sum |x_{mn}| u_{mn} \\
&= (\sup_{m+n \leq s} \alpha_{mn}^{-1}) \inf_{m+n \leq s} \alpha_{mn} \sum_{0 \leq m+n \leq s}\sum \frac{|x_{mn}| u_{mn}}{\inf_{m+n \leq s} \alpha_{mn}} \\
&\leq (\sup_{m+n \leq s} \alpha_{mn}^{-1}) \inf_{m+n \leq s} \alpha_{mn} \\
&= 1
\end{aligned}$$

since $x \in (\inf_{m+n \leq s} \alpha_{mn}) U_u$. Therefore, if

$$x \in (\inf_{m+n \leq s} \alpha_{mn}) [U_u \sqcap L_{s+1}],$$

we get $p_v(x) \leq 1$. Consequently

$$(\inf_{m+n \leq s} \alpha_{mn}) [U_u \sqcap L_{s+1}] \subset U_v$$

which amounts to

$$(**) \qquad \delta_{\frac{s(s+3)}{2}}(U_v, U_u) \geq \inf_{m+n \leq s} \alpha_{mn},$$

by Proposition 1.2.19. Since $\{\delta_s\}$ is decreasing, the required result follows from (*) and (**).

Lemma 3.3.3: If a bounded subset B of ℓ^{11} is relatively compact, then we have,

$$(+) \qquad \lim_{m+n\to\infty} \sup_{x\in B} \sum_{i+j\geq m+n}\sum |x_{ij}| = 0.$$

Proof: Suppose (+) is not true, then there exist increasing sequences $\{m_k\}$ and $\{n_k\}^k$ and $\varepsilon > 0$ such that

$$A_{I_k} \geq 2\varepsilon > 0; \ \forall\, k \geq 1$$

where

$$A_{I_k} = \sup_{x\in B} \sum_{i+j\geq I_k}\sum |x_{ij}| \text{ with } I_k = m_k+n_k.$$

Hence B contains a sequence $\{x^{I_k}\}$ such that

$$\sum_{i+j\geq I_k}\sum |x_{ij}^{I_k}| \geq \varepsilon, \ \forall\, k \geq 1$$

Let $p_1 = I_1$, then we can find $p_2 > p_1$, $p_2 = I_{k_o}$ for some k_o such that

$$\sum_{i+j\geq p_2}\sum |x_{ij}^{p_2}| \geq \varepsilon \quad\text{and}\quad \sum_{i+j\geq p_2}\sum |x_{ij}^{p_1}| < \varepsilon/2$$

Proceeding in this way we can find an increasing subsequence $\{p_k\}$ of $\{I_k\}$ such that

$$(*) \qquad \sum_{i+j\geq p_k}\sum |x_{ij}^{p_k}| \geq \varepsilon \ ; \quad \sum_{i+j\geq p_{k+1}}\sum |x_{ij}^{p_k}| < \varepsilon/2, \ \forall\, k \geq 1.$$

Therefore

$$(**)\qquad \sum_{i+j\geq 1}\sum |x_{ij}^{p_k} - x_{ij}^{p_t}| > \varepsilon/2,\ \forall\, k,t \geq 1,\ k\neq t,$$

for otherwise, say, for $k > t$ we have

$$\sum_{i+j\geq p_k}\sum |x_{ij}^{p_k} - x_{ij}^{p_t}| \leq \sum_{i+j\geq 1}\sum |x_{ij}^{p_k} - x_{ij}^{p_t}|$$

$$\leq \varepsilon/2.$$

Consequently

$$\sum_{i+j\geq p_k}\sum |x_{ij}^{p_k}| \leq \sum_{i+j\geq p_k}\sum |x_{ij}^{p_t}| + \varepsilon/2$$

$$\leq \sum_{i+j\geq p_{t+1}}\sum |x_{ij}^{p_t}| + \varepsilon/2$$

$$< \varepsilon$$

by using (*). Hence we get a contradiction. Therefore the sequence $\{x^{p_k}\}$ in B, for which (**) holds can not have any convergent subsequence and this again contradicts that B is relatively compact.

Lemma 3.3.4: Let X be an l.c. TVS and U_X, the neighbourhood system at origin consisting of all balanced, convex and closed sets. Suppose $A \subset X$ is bounded. Then A is precompact if and only if $\lim\limits_{m+n\to\infty} \delta_{m+n}(A,u) = 0$ for $u \in U_X$.

Proof. We first prove that A is precompact if

$$\lim_{m+n\to\infty} \delta_{m+n}(A,u) = 0.$$

Take $\varepsilon > 0$. One can find a large integer N such that

$$\delta_N(A,u) \leq \varepsilon/4$$

Hence there exists a finite set $\{y_1,\ldots,y_N\}$ of X and a positive number α such that

$$\alpha < \delta_N(A,u) + \varepsilon/4; \; A \subset \alpha u + \Gamma\{y_1,\ldots,y_N\}$$

by Proposition 1.2.16, where Γ is balanced, closed convex hull of the set $\{y_1,\ldots,y_N\}$. Hence

$$A \subset \frac{\varepsilon}{2} u + \Gamma\{y_1,\ldots,y_N\},$$

for u is balanced.

Let us now show that $\Gamma\{y_1,\ldots,y_N\}$ is precompact. Define

$$A_i = \{y_i\} \; ; \; 1 \leq i \leq N.$$

Then A_i's are compact and convex. Let

$$L = \{(\alpha_1,\ldots,\alpha_N) : \alpha_i \in \mathbb{K} \text{ and } \sum_{i=1}^{N} |\alpha_i| \leq 1\},$$

then L is a compact subset of $\mathbb{K}^{\mathbb{N}}$. Write

$$S = L \times A_1 \times \cdots \times A_N$$

It is clear by Tychonoff's theorem that S is compact in $\mathbb{K}^{\mathbb{N}} \times X^{\mathbb{N}}$. Now define a function $f: \mathbb{K}^{\mathbb{N}} \times X^{\mathbb{N}} \to X$ by

$$f(\alpha_1,\ldots,\alpha_N,x_1,\ldots,x_N) = \sum_{1 \leq i \leq N} \alpha_i x_i.$$

As f is continuous, $f(S)$ is compact. But $f(S)$ is nothing but the balanced closed convex hull of $\{y_1,\ldots,y_N\}$. So $\Gamma\{y_1,\ldots,y_N\}$ being compact, is precompact. Thus for finite number of elements $x_1,\ldots,x_M$ in $\Gamma\{y_1,\ldots,y_N\}$, we have

$$\Gamma\{y_1,\ldots,y_N\} \subset \bigcup_{i=1}^{M} \{x_i + \frac{\varepsilon}{2} u\}$$

consequently,

$$A \subset \frac{\varepsilon}{2}u + \bigcup_{i=1}^{M} \{x_i + \frac{\varepsilon}{2}u\} \subset \bigcup_{i=1}^{M} \{x_i + \varepsilon u\} .$$

This proves the precompactness of A.

Conversely, let A be precompact. Then by Proposition 1.2.17 $\delta_n(A,u) \to 0$ as $n \to \infty$. But we know $\delta_{m+n} \leq \delta_n$ for each $m \geq 0$, $n \geq 0$. Hence the required result follows.

Making use of the preceding results we prove

<u>Proposition</u> 3.3.5: Let $u, v \in P$ with $u_{mn} \leq v_{mn}$. Then

$$\delta_{m+n}(U_v, U_u) \to 0 \iff \{u_{mn}/v_{mn}\} \in c_{oo}$$

where $\alpha_{mn} = u_{mn}/v_{mn}$ is regarded to be zero whenever $v_{mn} = 0$.

<u>Proof</u>: Let $\{u_{mn}/v_{mn}\} \in c_{oo}$. By Proposition 3.3.2 we have

$$\delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u) \leq \sup_{m+n \geq s} \alpha_{mn}$$

$$\Longrightarrow \delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u) \to 0 \text{ as } s \to \infty .$$

Choose m and n so large that $m+n \geq \frac{(s+1)(s+2)}{2} \geq s$, and this proves the sufficiency part.

Conversely, let now $\delta_{m+n}(U_v, U_u) \to 0$. Define a map $T: \Lambda(P) \to \ell^{11}$ by

$$Tx = xu$$

where $u \in P$. Then

$$||Tx||_{\ell^{11}} = \sum_{m+n \geq 0}\sum |x_{mn}| u_{mn} = p_u(x)$$

Obviously

$$T(U_u) = \{\{x_{mn}u_{mn}\}; x \in \Lambda(P), \sum_{m+n \geq 0}\sum |x_{mn}| u_{mn} \leq 1\}$$

$$\Longrightarrow T(U_u) \subset U_{\ell^{11}}$$

where $U_{\ell^{11}}$ is the closed unit ball in ℓ^{11}. Therefore

$$\delta_{m+n}(T(U_v), U_{\ell^{11}}) \leq \delta_{m+n}(T(U_v), T(U_u)) \leq \delta_{m+n}(U_v, U_u) \to 0$$

by Proposition 1.2.14. So $T(U_v)$ is precompact by the Lemma 3.3.4 and hence relatively compact subset of ℓ^{11}. Define now

$$\theta^{st} = \alpha_{st}\, e^{st}\ ;\ s,t \geq 0$$

where α_{st} stands as in (3.3.1). Observe that $\theta^{st} \in U_{\ell^{11}}$. If $v_{st} = 0$, then $\theta^{st} = 0$ and so $\theta^{st} \in u.U_v$. If $v_{st} \neq 0$, then

$$\theta^{st} = \frac{u_{st}}{v_{st}}\, e^{st}.$$

But $e^{st}/v_{st} \in U_v$ and hence $\theta^{st} \in u.U_v$. Thus $\theta^{st} \in T(U_v)$ for all $s+t \geq 0$. Therefore

$$|\alpha_{st}| \leq \sup_{x \in T(U_v)} \sum_{m+n \geq s+t} \sum |x_{mn}|.$$

As $T(U_v)$ is relatively compact, by Lemma 3.3.3 we get

$$\lim_{s+t \to \infty} \sup_{x \in T(U_v)} \sum_{m+n \geq s+t} \sum |x_{mn}| = 0$$

$$\Longrightarrow \{\alpha_{st}\} \in c_{oo},$$

and hence

$$\left\{\frac{u_{mn}}{v_{mn}}\right\} \in c_{oo}.$$

Recalling the notations from Chapter I, section 2, we prove

<u>Proposition</u> 3.3.6: The canonical map $K_u^v : X_v \to X_u$ is precompact if and only if

$$(*) \qquad \delta_{m+n}(v,u) \to 0 \quad \text{as} \quad m+n \to \infty ,$$

where $u,v \in P$ with $u_{mn} \leq v_{mn}$ and $u_{mn} > 0$, $\forall$ m,n.

<u>Proof</u>. Let K_u^v be precompact. As the closed unit ball in X_v looks like $\{V+N_v : v \in V\}$, it follows that

$$K_u^v[V+N_v] = V+N_u \quad \text{is precompact in } X_u$$

$$\iff \delta_{m+n}(V+N_u, U+N_u) \to 0, \text{ by Proposition 1.2.17}$$

$$\iff \delta_{m+n}(v,u) \to 0, \text{ by Proposition 1.2.18.}$$

We are now ready to give the characterization of Schwartz property of the space $\Lambda(P)$ in the form of

__Theorem__ 3.3.7: The space $\Lambda(P)$ is Schwartz if and only if for each $u \in P$, there exists $v \in P$ with $u \leq v$ and such that $\{u_{mn}/v_{mn}\} \in c_{oo}$.

__Proof__: If $\Lambda(P)$ is Schwartz, then for each $u \in P$ there exists $v \in P$, $u \leq v$ and such that the map $K_u^v: (\Lambda(P))_v \to (\Lambda(P))_u$ is precompact. Hence from Propositions 3.3.5 and 3.3.6, we get

$$\{u_{mn}/v_{mn}\} \in c_{oo}$$

Conversely, suppose $\{u_{mn}/v_{mn}\} \in c_{oo}$. Once again, using the Propositions 3.3.5 and 3.3.6, we see that the map K_u^v is precompact, which in turn implies that $\Lambda(P)$ is Schwartz.

__Note__: The above result is known as the __Schock-Terzioglu Criterion__ for Schwartz property of the space $\Lambda(P)$.

We now consider two particular cases of the set P, as defined in

__Definition__ 3.3.8: Let the power set P satisfy the conditions

(i) Each $u \in P$ is non-decreasing, that is to say, $u_{mn} \leq u_{st}$ whenever $m+n \leq s+t$,

(ii) For each $a \in P$, there exists $b \in P$ such that $a_{mn}^2 \leq b_{mn}$, $\forall\, m,n$.

Then P is called a set of <u>infinite type</u> and the corresponding K-matrix space $\Lambda(P)$ is called a $G_{\infty\infty}$ -<u>space</u>.

<u>Definition</u> 3.3.9: If P satisfies the conditions

(i) Each $a \in P$ is non-increasing, that is to say, $a_{mn} \leq a_{st}$ whenever $m+n \geq s+t$.

(ii) For each $a \in P$, there exists $b \in P$ such that $a_{mn} \leq b^2_{mn}$, $\forall$ m,n.

Then P is said to be of <u>finite</u> <u>type</u> and the corresponding $\Lambda(P)$ is called a G_{11}-<u>space</u>.

The results characterizing Schwartzness of the $G_{\infty\infty}$- and G_{11}-spaces are respectively contained in the following two propositions:

<u>Proposition</u> 3.3.10: A $G_{\infty\infty}$-space is Schwartz if and only if there exists a $u \in P$ such that $\{1/u_{mn}\} \in c_{oo}$.

<u>Proof</u>: If the given space is Schwartz, then by Proposition 3.3.7, to each $a \in P$ there corresponds $u \in P$ such that $a \leq u$ and $\{a_{mn}/u_{mn}\} \in c_{oo}$; but due to the non-decreasing character of $\{a_{mn}\}$, it follows that

$$\frac{a_{oo}}{u_{mn}} \leq \frac{a_{mn}}{u_{mn}}$$

$$\Longrightarrow \{\frac{a_{oo}}{u_{mn}}\} \in c_{oo}$$

$$\Longrightarrow \{\frac{1}{u_{mn}}\} \in c_{oo} .$$

Conversely, let there be an $u \in P$ with $\{1/u_{mn}\} \in c_{oo}$. Choose an arbitrary a in P. Then one can find b and d in P such that

$$b_{mn} \geq \max \{a_{mn}, u_{mn}\} \text{ and } a_{mn}^2 \leq d_{mn}.$$

Let there be a g in P with $g_{mn} \geq \max \{b_{mn}, d_{mn}\}$. Hence

$$\frac{a_{mn}}{g_{mn}} \leq \frac{\sqrt{d_{mn}}}{g_{mn}} \leq \frac{1}{\sqrt{g_{mn}}} \leq \frac{1}{\sqrt{b_{mn}}} \leq \frac{1}{\sqrt{u_{mn}}} \to 0 \text{ as } m+n \to \infty.$$

Therefore

$$\{\frac{a_{mn}}{g_{mn}}\} \in c_{oo}.$$

So by Proposition 3.3.7, $\Lambda(P)$ is Schwartz.

If the hypothesis of the Schwartz character of a $G_{\infty\infty}$-space $\Lambda(P)$ is relaxed, we get

<u>Corollary</u> 3.3.11: If a $G_{\infty\infty}$-space $(\Lambda(P), T)$ is not Schwartz, then $\Lambda(P) = \ell^{11}$.

<u>Proof</u>: Let us observe that $\Lambda(P) \subset \ell^{11}$; indeed, for $u \in P$ with $u_{oo} > 0$, we have

$$\sum_{m+n \geq 0} \sum |x_{mn}| = \sum_{m+n \geq 0} \sum \frac{|x_{mn}| u_{mn}}{u_{mn}} \leq \frac{1}{u_{oo}} \Sigma\Sigma |x_{mn}| u_{mn}.$$

For proving the equality, assume that $\Lambda(P) \subsetneq \ell^{11}$. Then there exists an $x \in \ell^{11}$ such that $x \notin \Lambda(P)$, and so

$$(+) \qquad \sum_{m+n \geq 0} \sum |x_{mn}| < \infty;$$

also there exists an $u \in P$ for which the series

$$\sum_{m+n \geq 0}\sum |x_{mn}| u_{mn}$$

diverges. Hence for $N > 0$ we can determine integers R_N and S_N so that

$$\sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}| u_{mn} > N.$$

Hence we get

$$\begin{aligned} N &< \sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}| u_{mn} \\ &\leq u_{R_N S_N} \sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}| \\ &\leq u_{R_N S_N} \sum_{m+n \geq 0}\sum |x_{mn}|. \end{aligned}$$

So by (+) we have

$$\frac{1}{u_{R_N S_N}} \leq \frac{1}{N} \Big(\sum_{m+n \geq 0}\sum |x_{mn}| \Big) \to 0$$

as $N \to \infty$. Since $\{u_{mn}\}$ is non-decreasing $\{1/u_{mn}\} \in c_{oo}$. But this contradicts the fact that $\Lambda(P)$ is not Schwartz. Hence the result follows.

<u>Proposition</u> 3.3.12: A G_{11}-space $(\Lambda(P),T)$ is Schwartz if and only if $P \subset c_{oo}$.

<u>Proof</u>: Let $P \subset c_{oo}$. Take $u \in P$, so by the G_{11}-character, we can find a $v \in P$ with $u_{mn} \leq v_{mn}^2$, for $m,n \geq 0$. But $v \in P \subset c_{oo}$ and so

$$\frac{u_{mn}}{v_{mn}} \leq v_{mn} \to 0 \quad \text{as} \quad m+n \to \infty$$

Therefore, $\{u_{mn}/v_{mn}\} \in c_{oo}$ and hence $\Lambda(P)$ is Schwartz by Proposition 3.3.7.

Conversely, let $\Lambda(P)$ be Schwartz and assume $u \in P$. Then we can find some $v \in P$ such that

$$\{\frac{u_{mn}}{v_{mn}}\} \in c_{oo}.$$

But members of P are non-increasing

$$\Longrightarrow \frac{u_{mn}}{v_{mn}} \geq \frac{u_{mn}}{v_{oo}}$$

$$\Longrightarrow \{u_{mn}/v_{oo}\} \in c_{oo}$$

$$\Longrightarrow \{u_{mn}\} \in c_{oo}$$

$$\Longrightarrow P \subset c_{oo} ,$$

and the result is proved.

3.4. Bi-dimetrical Dimension

In the last section, we use the notion of Kolmogoroff's diameters for the neighbourhoods of an arbitrary l.c.TVS (X,T) for characterizing its Schwartz property. Indeed, the Kolmogoroff's diameters of neighbourhoods of zero in an l.c. TVS (X,T) yield a matrix space introduced in

Definition 3.4.1: For the fundamental neighbourhood system U_X of an l.c. TVS (X,T), the space defined by

$$\Delta_{U_X}(X) = \{\alpha = (\alpha_{mn}) : \text{ to each } u \in U_X \text{ there exists } v \in U_X \text{ with } v \prec u \text{ and } \lim_{m+n\to\infty} |\alpha_{mn}|\, \delta_{m+n}(v,u) = 0\}$$

is called the *bi-dimetrical* dimension of X.

To begin with, we have

<u>Proposition</u> 3.4.2: If U'_X is another neighbourhood system at origin besides U_X, then

$$\Delta_{U_X}(X) = \Delta_{U'_X}(X)$$

<u>Proof</u>: Let $x \in \Delta_{U_X}(X)$. Since neighbourhood systems are equivalent, if $u' \in U'_X$, then there exists $u \in U_X$ with $u \subset u'$. So then for $v \in U_X$ with $v < u$ we get

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0.$$

Given $v \in U_X$ there exist $v' \in U'_X$ such that $v' \subset v$. Thus,

$$v' \subset v < u \subset u'$$

$$\Longrightarrow \delta_{m+n}(v',u') \leq \delta_{m+n}(v',u) \leq \delta_{m+n}(v,u)$$

$$\Longrightarrow |x_{mn}|\delta_{m+n}(v',u') \leq |x_{mn}|\delta_{m+n}(v,u) \to 0 \text{ as } m+n \to \infty$$

So $x \in \Delta_{U'_X}(X)$ and hence $\Delta_{U_X}(X) \subset \Delta_{U'_X}(X)$.

Now let $x \in \Delta_{U'_X}(X)$. Take $u \in U_X$, then there exists $u' \in U'_X$ such that $u' \subset u$. So, then for $v' \in U'_X$ with $v' < u'$ we have

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v',u') = 0$$

But for $v' \in U'_X$, there exists $v \in U_X$ such that $v \subset v'$. So

$$v \subset v' < u' \subset u$$

$$\Longrightarrow \delta_{m+n}(v,u) \leq \delta_{m+n}(v,u') \leq \delta_{m+n}(v',u')$$

$$\Longrightarrow |x_{mn}|\delta_{m+n}(v,u) \leq |x_{mn}|\delta_{m+n}(v',u') \to 0 \quad \text{as} \quad m+n \to \infty$$

So $x \in \Delta_{U_X}(X)$, and hence $\Delta_{U_X}(X) = \Delta_{U'_X}(X)$.

Remark 3.4.3: Since $\Delta_{U_X}(X)$ is independent of the choice of U_X, we would prefer to use the notation $\Delta(X)$ for $\Delta_{U_X}(X)$

The invariance of bi-dimetrical dimension under homeomorphism is shown in

Proposition 3.4.4: If two locally convex spaces (X,T_1) and (Y,T_2) are topologically isomerphic by the map A, then we get

$$\Delta(X) = \Delta(Y)$$

Proof: By Proposition 1.2.14, we have

$$\delta_{m+n}(v,u) \geq \delta_{m+n}(Av,Au) \geq \delta_{m+n}(A^{-1}(Av), A^{-1}(Au))$$
$$= \delta_{m+n}(v,u)$$

where $u,v \in U_X$ with $v < u$. Then the above inequality implies that

$$(*) \qquad \delta_{m+n}(v,u) = \delta_{m+n}(Av,Au)$$

Let

$$U^* = \{Au; u \in U_X\}$$

From $(*)$ we find that $\Delta(X) = \Delta_{U^*}(Y)$. But by Remark 3.4.3, $\Delta_{U^*}(Y) = \Delta(Y)$. Hence $\Delta(X) = \Delta(Y)$.

The above result can also be derived from a more general result contained in

Theorem 3.4.5: (Permanance Theorem). Let X, Y be two l.c. TVS. Suppose $A : X \to Y$ is linear, continuous and almost open, then we have

$$\Delta(X) \subset \Delta(Y)$$

Proof: Let $u^* \in U_Y$. So $A^{-1}(u^*) \supset u$, for some $u \in U_X$ by our assumption of continuity on the linear map A.

Let $x \in \Delta(X)$, so for $u \in U_X$ there exists $v \in U_X$ with $v < u$ and such that

$$\lim_{m+n \to \infty} |x_{mn}| \delta_{m+n}(v, u) = 0$$

we now proceed to find v^* in U_Y such that $v^* < u^*$ with

$$\lim_{m+n \to \infty} |x_{mn}| \delta_{m+n}(v^*, u^*) = 0$$

Let us define $v^* = \overline{Av}$. So $v^* \in U_Y$ for A is almost open. Then we have

$$\lim_{m+n \to \infty} |x_{mn}| \delta_{m+n}(v^*, u^*) \leq \lim_{m+n \to \infty} |x_{mn}| \delta_{m+n}(Av, Au)$$

$$\leq \lim_{m+n} |x_{mn}| \delta_{m+n}(v, u),$$

and hence $x = \{x_{mn}\} \in \Delta(Y)$, consequently

$$\Delta(X) \subset \Delta(Y).$$

In the rest of this section, we try to find the relationship of the bi-dimetrical dimension of an l.c. TVS (X,T) with the known matrix spaces. In the case of weak topology, we have the following most general result:

Theorem 3.4.6: Let (X,Y) be a dual system. Then

$$\Delta(X,\sigma(X,Y)) = \Omega$$

Proof: Clearly, $\Delta(X,\sigma(X,Y)) \subset \Omega$. So let u be a neighbourhood of origin in X with respect to $\sigma(X,Y)$. Then there exists $\varepsilon > 0$ such that $\varepsilon u_s \subset u$, where for $y_1, \ldots, y_s \in Y$,

$$u_s = \{x \in X: \ |\langle x, y_i\rangle| \leq 1, \ 1 \leq i \leq s\}.$$

Let

$$N_s = \{x \in X: \ \langle x, y_i\rangle = 0, \ 1 \leq i \leq s\}$$

So N_s is a $\sigma(X,Y)$-closed subspace of X. Then by Definition 1.2.3 we have $\dim(X/N_s) \leq s$. There exists a closed subspace L_s such that

$$X = N_s \oplus L_s$$

Now observe that

$$\dim(L_s) = \dim(X/N_s) \leq s.$$

If $\rho < 1$, then

$$u_s \subset N_s + L_s \subset \rho u_s + L_s$$

$$\Longrightarrow \delta_s(u_s, u_s) \leq \rho$$

Since ρ is arbitrary, allow $\rho \to 0$. Then we get

$$\delta_s(u_s,u_s) = 0$$

$$\Longrightarrow \delta_{m+n}(u_s,u_s) = 0, \; m+n \geq s.$$

Because $\varepsilon u_s \subset u$, we have the inequality

$$\delta_{m+n}(u_s,u) \leq \delta_{m+n}(u_s,u_s)$$

$$\Longrightarrow \delta_{m+n}(u_s,u) = 0, \; m+n \geq s$$

Now take any $x \in \omega$. Then we have

$$|x_{mn}|\delta_{m+n}(u_s,u) = 0, \quad m+n \geq s$$

$$\Longrightarrow \lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(u_s,u) = 0,$$

and so $x \in \Delta(X,\sigma(X,Y))$, there by proving the required result.

For an arbitrary locally convex topology on a vector space X, we prove

Proposition 3.4.7: For an l.c. TVS (X,T), $c_{oo} \subset \Delta(X)$.

Proof: Let $x = \{x_{mn}\} \in c_{oo}$. Hence for u in U_X there exists a large integer N such that $x_{mn} \in u$ for all $m+n \geq N$.

For proving the result, it is enough to show that if $v \in U_X$ with $v < u$, then

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0$$

Since $\delta_o(v,u) \leq \rho$, where ρ comes from $v \subset \rho u$ and

$$\delta_o(v,u) \geq \delta_1(v,u) \geq \delta_2(v,u) \geq \ldots,$$

we find

$$\delta_{m+n}(v,u) \leq \rho, \quad \forall\, m,n,\ m+n \geq N$$

$$\Longrightarrow \lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0,$$

and hence the result is proved.

Proper inclusion of c_{oo} in $\Delta(X)$, leads to a characterization of the Schwartz character of the space as shown in the main result of this section, namely.

<u>Theorem</u> 3.4.8: For an l.c. TVS (X,T), the following statements are equivalent:

(i) X is Schwartz,

(ii) $\ell^{\omega\omega} \subset \Delta(X)$,

(iii) $c_{oo} \subsetneq \Delta(X)$.

<u>Proof</u>: (i) $\Longrightarrow$ (ii). Let X be Schwartz and assume that $x \in \ell^{\infty\infty}$. So to each $u \in U_X$, there exists a $u \in U_X$, $v \prec u$ such that

$$\delta_{m+n}(v,u) \to 0 \quad \text{as} \quad m+n \to \infty$$

and also there exists $k > 0$ such that $|x_{mn}| \leq k$ for all $m,n \geq 0$. Thus

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0,$$

which gives $x \in \Lambda(X)$ and hence $\ell^{\infty\infty} \subset \Delta(X)$.

(ii) ==> (iii). Assume that $\ell^{\infty\infty} \subset \Delta(X)$, but we know $c_{oo} \subsetneqq \ell^{\infty\infty}$ and hence $c_{oo} \subsetneqq \Delta(X)$.

(iii) ==> (i). Let $c_{oo} \subsetneqq \Delta(X)$, then there exist $x \in \Delta(X)$ but $x_{mn} \not\to 0$ as $m+n \to \infty$. So there exists sequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$\inf |x_{m_k n_k}| = \eta > 0.$$

Take $u \in U_X$, then there exists $v \in U_X$ with $v \prec u$ and such that

$$(*) \qquad \lim_{m+n \to \infty} |x_{mn}| \delta_{m+n}(v,u) = 0$$

But we know

$$\eta \delta_{m_k+n_k}(v,u) \leq |x_{m_k n_k}| \, \delta_{m_k+n_k}(v,u)$$

$$\Longrightarrow \eta \, \delta_{m_k+n_k}(v,u) \to 0$$

by (*). So we get $\delta_{m+n}(v,u) \to 0$, and hence X is Schwartz by Lemma 3.3.5 and Proposition 3.3.6.

<u>Corollary</u> 3.4.9: Let $A: X \to Y$ be a continuous linear and almost open map. Suppose X is Schwartz, then Y is Schwartz.

<u>Proof</u>: Under the given hypothesis we have

$$\Delta(X) \subset \Delta(Y)$$

by Theorem 3.4.5. But X is Schwartz, so we get

$$\ell^{\infty\infty} \subset \Delta(X)$$

$$\Longrightarrow \ell^{\infty\infty} \subset \Delta(Y).$$

Therefore Y is Schwartz by Theorem 3.4.8.

Chapter 4

Matrix Structure of Spaces of Entire Functions

4.1 Introduction

This chapter is a continuation of our study on the matrix space $\delta^{\alpha,\beta}$ introduced in Section 2 of Chapter 3, which, indeed, envelops the class of all entire functions of two variables as a particular case. In this chapter too, we assume throughout that the sequences α and β satisfy the condition (3.2.9). In the beginning of this chapter we find the K-dual of $\delta^{\alpha,\beta}$ and characterize its bounded sets relative to the weak topology. Then we proceed to characterizing the continuous linear functionals on this class as well as a subclass of this class, which correspond to subspaces of δ having order zero and type one. Besides, we also pay attention to investigate structural properties of this subspace.

4.2 On the Space $\delta^{\alpha,\beta}$

Let us recall the sequences α,β and the space $\delta^{\alpha,\beta}$, from Section 2 of the previous chapter and introduce the following matrix space

$$(4.2.1) \quad d^{\alpha,\beta} = \{x \in \Omega : \sup \{ |x_{00}| ; |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, \forall\, m+n \neq 0 \} < \infty \}$$

Then we have

Proposition 4.2.2: $(\delta^{\alpha,\beta})^x = d^{\alpha,\beta}$ and $(d^{\alpha\beta})^x = \delta^{\alpha\beta}$.

Proof: For showing the first equality, consider $x \in d^{\alpha,\beta}$.
Then there exists $M > 0$ with

$$|x_{ij}| < M^{\alpha_i+\beta_j}, \quad i+j \geq 0$$

One may choose $\varepsilon > 0$ so small that $\varepsilon M < 1$. If $y \in \delta^{\alpha,\beta}$, then we get

$$|y_{ij}| \leq \varepsilon^{\alpha_i+\beta_j}, \quad i+j \geq i_o = i_o(\varepsilon)$$

Therefore

$$\sum_{i+j \geq 0}\sum |x_{ij}y_{ij}| = \sum_{0 \leq i+j < i_o}\sum |x_{ij}y_{ij}| + \sum_{i+j \geq i_o}\sum |x_{ij}y_{ij}|$$

$$\leq \sum_{0 \leq i+j < i_o}\sum |x_{ij}y_{ij}| + \sum_{i+j \geq i_o}\sum (\varepsilon M)^{\alpha_i+\beta_j} < \infty ;$$

$$\Longrightarrow d^{\alpha,\beta} \subset (\delta^{\alpha,\beta})^x$$

On the other hand, let $x \in (\delta^{\alpha,\beta})^x$ but $x \notin d^{\alpha,\beta}$.
Hence there exists increasing sequences $\{m_k\}$ and $\{n_k\}$ of integers such that

$$|x_{m_k n_k}| > k^{2(\alpha_{m_k}+\beta_{n_k})}, \quad k \geq 1$$

If

$$y_{mn} = \begin{cases} k^{-(\alpha_{m_k}+\beta_{n_k})} ; & m=m_k,\ n=n_k ; \\ 0 , & \text{elsewhere.} \end{cases}$$

Then $y = \{y_m\} \in \delta^{\alpha,\beta}$, but $\sum\sum_{i+j\geq 0} |x_{ij}y_{ij}| = \infty$. Thus $x \notin (\delta^{\alpha,\beta})^x$ which is a contradiction. Therefore $(\delta^{\alpha,\beta})^x \subset d^{\alpha,\beta}$. So we get $(\delta^{\alpha,\beta})^x = d^{\alpha,\beta}$.

For the other equality, observe that

$$\delta^{\alpha,\beta} \subset (\delta^{\alpha,\beta})^{xx} = (d^{\alpha,\beta})^x.$$

On the other hand, let $x \in (d^{\alpha,\beta})^x$ but $x \notin \delta^{\alpha,\beta}$. Hence there exist $\varepsilon > 0$ and increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|x_{m_k n_k}| \geq \varepsilon^{\alpha_{m_k}+\beta_{n_k}}, \ \forall\, k \geq 1.$$

If $y \in \Omega$, is given by

$$y_{mn} = \begin{cases} \varepsilon^{-(\alpha_{m_k}+\beta_{n_k})}, & m=m_k,\ n=n_k; \\ 0, & \text{elsewhere} \end{cases}$$

Then $y \in d^{\alpha,\beta}$, but $\sum\sum_{i+j\geq 0} |x_{ij}y_{ij}| = \infty$. Thus $x \notin (d^{\alpha,\beta})^x$ which is a contradiction. So $(d^{\alpha,\beta})^x \subset \delta^{\alpha,\beta}$. Consequently $(d^{\alpha,\beta})^x = \delta^{\alpha,\beta}$.

<u>Corollary</u> 4.2.3: The spaces $\delta^{\alpha,\beta}$ and $d^{\alpha,\beta}$ are perfect.

Next, we prove

<u>Proposition</u> 4.2.4: A subset B of $\delta^{\alpha,\beta}$ is $\sigma(\delta^{\alpha,\beta}, d^{\alpha,\beta})$ bounded if and only if the following hold:

(i) B is $\sigma(\delta^{\alpha,\beta}, \Phi)$ bounded

(ii) For each $\varepsilon > 0$, there exists $N_o \equiv N_o(\varepsilon)$ such that $|x_{mn}| \leq \varepsilon^{\alpha_m+\beta_n}$, $\forall\, m+n \geq N_o$, $\forall\, x \in B$.

Proof: Let (i) and (ii) be true. Take $y \in d^{\alpha,\beta}$, so for a suitable $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$|y_{mn}| \leq (2\varepsilon)^{-(\alpha_m+\beta_n)}, \ \forall\, m+n \geq N$$

Let $N^* = \max\{N_o, N\}$, then we have

$$\sum_{i+j \geq N^*}\sum |x_{ij}y_{ij}| \leq \sum_{i+j \geq N^*}\sum \frac{1}{2^{\alpha_i+\beta_j}} < \sum_{i+j \geq 1}\sum \frac{1}{2^{\alpha_i+\beta_j}}, \ \forall\, x \in B$$

Since $z = \sum_{i+j \leq N^*}\sum y_{ij}e^{ij}$ is in Φ, by (1) there exists a constant $M \equiv M(N^*) > 0$ such that

$$\sum_{0 \leq i+j \leq N^*}\sum |x_{ij}y_{ij}| \leq M$$

$$\Longrightarrow \sum_{i+j \geq 0}\sum |x_{ij}y_{ij}| \leq M + \sum_{i+j \geq N^*}\sum \frac{1}{2^{\alpha_i+\beta_j}}$$

$$\leq M + \sum_{i+j \geq 0}\sum \frac{1}{2^{\alpha_i+\beta_j}}$$

$$< \infty$$

Conversely, let B be $\sigma(\delta^{\alpha,\beta}, d^{\alpha,\beta})$ bounded, then it is enough to show that (ii) holds. Suppose (ii) is not true, then for some $\varepsilon > 0$, there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ and a sequence $x^{(k)}$ in B such that

$$|x^{(k)}_{m_k n_k}| > \varepsilon^{\alpha_{m_k}+\beta_{n_k}}, \ \forall\, k \geq 1$$

Define $y \in \Omega$ by.

$$y_{mn} = \begin{cases} \dfrac{1}{|x^{(k)}_{m_k n_k}|} \, \alpha_{m_k}^{1+\frac{\beta_{n_k}}{\alpha_{m_k}}} \, \beta_{n_k}^{1+\frac{\alpha_{m_k}}{\beta_{n_k}}} & ; \; m = m_k, \; n = n_k \; ; \\ 0 \; , & \text{elsewhere} \end{cases}$$

Then we have for $m = m_k$ and $n = n_k$

$$|y_{m_k n_k}|^{\frac{1}{\alpha_{m_k}+\beta_{n_k}}} < \frac{1}{\varepsilon} \, \alpha_{m_k}^{\frac{1}{\alpha_{m_k}}} \, \beta_{n_k}^{\frac{1}{\beta_{n_k}}}$$

$$\to \frac{1}{\varepsilon} \quad \text{as} \quad k \to \infty \; .$$

Hence $y \in d^{\alpha,\beta}$. Consider now

$$p_y(x^{(k)}) = \sum_{m+n \geq 0} \sum |x^{(k)}_{mn} \, y_{mn}|$$

$$\geq |x^{(k)}_{m_k n_k} \, y_{m_k n_k}|$$

$$= \alpha_{m_k}^{1+\frac{\beta_{n_k}}{\alpha_{m_k}}} \, , \; \beta_{n_k}^{1+\frac{\alpha_{m_k}}{\beta_{n_k}}}$$

Therefore B is not $\eta(\delta^{\alpha,\beta}, d^{\alpha,\beta})$ bounded, and hence not $\sigma(\delta^{\alpha,\beta}, d^{\alpha,\beta})$ bounded, because $\delta^{\alpha,\beta}$ is perfect. This is a contradiction. So (ii) must hold.

For our next result, we need

<u>Lemma</u> 4.2.5: Consider a sequence $\{a_{mn} ; m,n \geq 0\}$, satisfying

$$\lim_{m+n \to \infty} |a_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0 .$$

Then the series

$$(+) \qquad \sum_{m+n \geq 0}\sum c_{mn} a_{mn}$$

converges if and only if

$$\{ |c_{00}| ; \ |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, \ m+n > 0 \}$$

is bounded.

<u>Proof:</u> Assume first of all that, there exists a positive constant M, such that

$$|c_{00}| \leq M, \ |c_{mn}| \leq M^{\alpha_m+\beta_n}, \ m,n \geq 0, \ m+n \neq 0$$

By hypothesis, we get an integer N such that

$$|a_{mn}| \leq \left(\frac{1}{2M}\right)^{\alpha_m+\beta_n}, \ m+n \geq N,$$

and hence

$$|a_{mn}c_{mn}| \leq 2^{-(\alpha_m+\beta_n)}, \ m+n \geq N.$$

Therefore, for m+n $\geq$ N,

$$\sum_{m+n \geq 0}\sum |a_{mn}c_{mn}| \leq \sum_{0 \leq m+n \leq N-1}\sum |a_{mn}c_{mn}| + \sum_{m+n \geq N}\sum \frac{1}{2^{\alpha_m+\beta_n}}$$

$$\leq \sum_{0 \leq m+n \leq N-1}\sum |a_{mn}c_{mn}| + \sum_{m+n \geq 0}\sum \frac{1}{2^{\alpha_m+\beta_n}} < \infty$$

So $\sum_{m+n \geq 0}\sum a_{mn}c_{mn}$ converges.

To show the necessity, assume that the series (+) is convergent, but the sequence

$$\{|c_{00}|;\ |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}},\ m+n > 0\}$$

is unbounded. Therefore, there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|c_{m_k n_k}| \geq k^{\alpha_{m_k}+\beta_{n_k}},\ k \geq 1.$$

Now define a new sequence $a = (a_{mn})$ by

$$a_{mn} = \begin{cases} k^{-(\alpha_{m_k}+\beta_{n_k})}, & m = m_k, n = n_k; \\ 0, \text{ elsewhere} \end{cases}$$

$$\Longrightarrow |a_{m_k n_k}\ c_{m_k n_k}| \geq 1,\ \forall\ k \geq 1.$$

Hence $\sum\sum_{m+n \geq 0} |a_{nm} c_{mn}|$ does not converge, though

$$\lim_{m+n\to\infty} |a_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0.$$

The contradiction arrived at, completes the proof.

Concerning the representation of continuous linear functionals on $\delta^{\alpha,\beta}$, we prove

<u>Theorem</u> 4.2.5: Consider $\delta^{\alpha,\beta}$ equipped with either T_P or $T_{\alpha,\beta}$. Then every continuous linear functional Ψ on $\delta^{\alpha,\beta}$ is of the form

$$(4.2.7) \qquad \Psi(x) = \sum_{m+n \geq 0}\sum x_{mn} c_{mn} \,,$$

where

$$(4.2.8) \qquad \{ |c_{oo}| ; |c_{mn}|^{\frac{1}{\alpha_m + \beta_n}} , m+n > 0 \}$$

is bounded.

Moreover, for any double sequence, $\{c_{mn}\}$ satisfying (4.2.8), the map $\Psi : \delta^{\alpha,\beta} \to \mathbb{C}$, whose value of x is given by (4.2.7), represents a continuous linear functional.

<u>Proof</u>: Let $\Psi \in (\delta^{\alpha,\beta})^*$. Now define

$$c_{mn} = \Psi(e^{mn}).$$

For any element x in $\delta^{\alpha,\beta}$, the s-th plane section of x is given by

$$x^{(s)} = \sum_{0 \leq m+n \leq s}\sum x_{mn} e^{mn}.$$

For $a = \{a_{mn}\}$ in P, the series $\sum\limits_{m+n \geq 0}\sum x_{mn} a_{mn}$ converges, therefore.

$$x^{(s)} \to x \text{ in } T_P$$

$$\Longleftrightarrow \quad x^{(s)} \to x \text{ in } T_{\alpha,\beta}$$

by Proposition 3.2.10. Hence

$$x = \sum_{m+n \geq 0}\sum x_{mn} e^{mn} .$$

Consequently

$$\Psi(x) = \lim_{s\to\infty} \Psi(x^{(s)})$$

$$= \lim_{s\to\infty} \sum_{0\leq m+n\leq s}\sum x_{mn}\ \Psi(e^{mn})$$

$$= \sum_{m+n\geq 0}\sum x_{mn}c_{mn}.$$

But

$$\lim_{m+n} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0,$$

and so

$$\{|c_{oo}| \ ; \ |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}} : m+n > 0\}$$

is bounded by Lemma 4.2.5.

Conversely, let Ψ be as mentioned in the hypothesis. Then it is enough to show that Ψ is continuous, linearity being clear from the definition. So let $x^p \to 0$ in T_p. Take $\varepsilon > 0$ arbitrarily. By the condition (3.2.9) we can find an integer N such that

(*) $$\alpha_n-\alpha_{n-1}\geq h; \ \beta_n- \beta_{n-1} \geq k, \ \forall\ n \geq N$$

Write

$$M = \sup \{|c_{oo}|, |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, m+n \neq 0\}.$$

Choose $\eta > 0$ so small that $\eta M < 1$ and

$$\eta M+\frac{(\eta M)^{\alpha_1+\beta_1}}{[1-(\eta M)^h][1-(\eta M)^k]}[N(1-(\eta M)^h)+(\eta M)^h][N(1-(\eta M)^k)+(\eta M)^k]<\varepsilon$$

Then there exists an integer Q such that

$$|x^p_{oo}| \; ; \; |x^p_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \eta, \quad m+n\neq 0, \; p\geq Q$$

Let us observe that, for $p \geq Q$

$$|\Psi(x^p)| \leq |x^p_{oo}c_{oo}| + \sum_{m+n\geq 1}\sum |x^p_{mn}c_{mn}|$$

$$\leq \eta M + \sum_{m+n\geq 1}\sum (\eta M)^{\alpha_m}(\eta M)^{\beta_n}.$$

But, using (*), we get

$$\sum_{m\geq 1}(\eta M)^{\alpha_m} = \sum_{m=1}^{N-1}(\eta M)^{\alpha_m} + \sum_{m\geq N}(\eta M)^{\alpha_m}$$

$$\leq \sum_{m=1}^{N-1}(\eta M)^{\alpha_m} + (\eta M)^{\alpha_N} + (\eta M)^{\alpha_N+h} + (\eta M)^{\alpha_N+2h} + \dots$$

$$= \sum_{m=1}^{N-1}(\eta M)^{\alpha_m} + \frac{(\eta M)^{\alpha_N}}{1-(\eta M)^h}$$

$$\leq (N-1)(\eta M)^{\alpha_1} + \frac{(\eta M)^{\alpha_1}}{[1-(\eta M)^h]}$$

Proceeding similarly, we also have

$$\sum_{n\geq 1}(\eta M)^{\beta_n} \leq (N-1)(\eta M)^{\beta_1} + \frac{(\eta M)^{\beta_1}}{[1-(\eta M)^k]}\;.$$

Hence, for $p \geq Q$

$$|\Psi(x^p)| \leq \eta M + [(N-1)(\eta M)^{\alpha_1} + \frac{(\eta M)^{\alpha_1}}{[1-(\eta M)^h]}][(N-1)(\eta M)^{\beta_1} + \frac{(\eta M)^{\beta_1}}{[1-(\eta M)^k]}]$$

$$= \eta M + \frac{(\eta M)^{\alpha_1+\beta_1}}{[1-(\eta M)^h][1-(\eta M)^k]}[N(1-(\eta M)^h)+(\eta M)^h][N(1-(\eta M)^k)+(\eta M)^k]$$

$$< \varepsilon .$$

Thus Ψ is continuous. This completely establishes the result.

Making use of the above result, we prove

Proposition 4.2.9: The spaces $((\delta^{\alpha,\beta})^*, \beta((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta}))$ and $(d^{\alpha,\beta}, \beta(d^{\alpha,\beta}, \delta^{\alpha,\beta}))$ are topologically isomorphic.

Proof: In view of Theorem 4.2.6, we can define a natural map $R: (\delta^{\alpha,\beta})^* \to d^{\alpha,\beta}$ by the relation

$$R(\Psi) = [c_{mn}], \quad \Psi \in (\delta^{\alpha,\beta})^*$$

where $[c_{mn}] \in d^{\alpha,\beta}$, with

$$\Psi(x) = \sum\sum_{m+n\geq 0} x_{mn}c_{mn}, \quad x \in \delta^{\alpha,\beta}.$$

Then R is clearly a 1-1, onto linear map.

To prove the result, we first show the topological isomorphic character of R with respect to the weak topologies $\sigma((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta})$ and $\sigma(d^{\alpha,\beta}, \delta^{\alpha,\beta})$. Therefore, consider a net $\{\Psi^\gamma\}$ in $(\delta^{\alpha,\beta})^*$ such that $\Psi^\gamma \to 0$ in $\sigma((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta})$. Let

$$\Psi^\gamma(x) = \sum\sum_{m+n\geq 0} x_{mn}\, c^\gamma_{mn}$$

for $[c_{mn}^{\gamma}] \in d^{\alpha,\beta}$. Then for $x \in \delta^{\alpha,\beta}$ with $x = [x_{mn}]$ and for $\varepsilon > 0$, there exists $\gamma_0 \equiv \gamma_0(\varepsilon)$ such that

$$|\sum_{m+n \geq 0}\sum x_{mn}c_{mn}^{\gamma}| < \varepsilon, \ \gamma \geq \gamma_0$$

$$\Longrightarrow \ [c_{mn}^{\gamma}] \to 0 \ \text{ in } \ \sigma(d^{\alpha,\beta},\delta^{\alpha,\beta})$$

Thus R is $\sigma((\delta^{\alpha,\beta})^*,\delta^{\alpha,\beta}) - \sigma(d^{\alpha,\beta},\delta^{\alpha,\beta})$ continuous. Retracing back the above implications, we get the $\sigma(d^{\alpha,\beta},\delta^{\alpha,\beta}) - \sigma((\delta^{\alpha,\beta})^*,\delta^{\alpha,\beta})$ continuity of R^{-1}.

The $\beta((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta}) - \beta(d^{\alpha,\beta},\delta^{\alpha,\beta})$ continuity of R is a consequence of Proposition 1.2.5 and the following equality, namely,

$$\begin{aligned}
<Ix,\Psi> &= <x,\Psi'> \\
&= \sum_{m+n>0}\sum x_{mn}c_{mn} \\
&= <x,[c_{mn}]> \\
&= <x,R(\Psi)>
\end{aligned}$$

where I is the identity map from $\delta^{\alpha,\beta}$ to itself. Similarly, the Proposition 1.2.5 and the equality

$$\begin{aligned}
<Ix, [c_{mn}]> &= <x,[c_{mn}]>, \\
&= \sum_{m+n>0}\sum x_{mn}c_{mn} \\
&= <x,\Psi> \\
&= <x,R^{-1}[c_{mn}]>,
\end{aligned}$$

gives the $\beta(d^{\alpha,\beta},\delta^{\alpha,\beta}) - \beta((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta})$ continuty of R^{-1}. Therefore R is the required topological isomorphism between the spaces $((\delta^{\alpha,\beta})^*, \beta((\delta^{\alpha,\beta})^*, \delta^{\alpha,\beta}))$ and $(d^{\alpha,\beta},\beta(d^{\alpha,\beta},\delta^{\alpha,\beta}))$. This establishes the result.

4.3 Certain Subspaces of a Fréchet Space

We consider here a special type of matrix spaces, which in particular, envelop a class of entire functions of two variables. The purpose of this section is to study the topological properties of these spaces and continuous linear functionals thereon.

To be precise, let $p \equiv \{p_m, m \geq 0\}$ and $q \equiv \{q_n, n \geq 0\}$ be two bounded sequence of real numbers such that $p_m > 0$, $q_n > 0$ and $\{p_m^{-1}\}$ and $\{q_n^{-1}\}$ satisfy Mandelbrojt restriction, namely, if $\mu_m = \frac{1}{p_m}$ and $\lambda_n = \frac{1}{q_n}$, then

$$\text{(4.3.1)} \qquad \underline{\lim}\ (\mu_m - \mu_{m-1}) = h > 0; \ \underline{\lim}\ (\lambda_n - \lambda_{n-1}) = k > 0.$$

For $M = \max\{1, \sup_m p_m\}$, $N = \max\{1, \sup_n q_n\}$ and $\beta \geq 0$, we introduce the spaces

$$\Gamma_\beta(p,q) = \{a \in \Omega : \overline{\lim}(m+n)|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \beta\},\ \beta > 0$$

and

$$\Gamma_0(p,q) = \{a \in \Omega : |(m+n)!\,a_{mn}|^{\frac{1}{p_m^{-1}+q_n^{-1}}} \to 0 \text{ as } m+n \to \infty\}.$$

Then, the spaces $\Gamma_\beta(p,q)$, $\beta \geq 0$, are clearly vector spaces with respect to usual pointwise addition and scalar multiplication. However, the topological behaviour of the spaces $\Gamma_\beta(p,q)$, $\beta > 0$ and $\Gamma_0(p,q)$ don't seem to be analogous and so we bifurcate this study in the following two subsections.

<u>The Spaces</u> $\Gamma_\beta(p,q)$, $\beta > 0$

For $\varepsilon > 0$, we define a positive real valued function $||\cdot,\beta+\varepsilon||$ on $\Gamma_\beta(p,q)$ as follows

$$(4.3.2) \qquad ||a,\beta+\varepsilon|| = |a_{00}| + \sum_{m+n\geq 1}\sum |a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, a \in \Gamma_\beta(p,q).$$

At the outset, we have

<u>Proposition</u> 4.3.8: The collection $D_\beta = \{||\cdot,\beta+\varepsilon||, \varepsilon > 0\}$ defines a decreasing family of norms on the space $\Gamma_\beta(p,q)$ and $(\Gamma_\beta(p,q), T_\beta)$, where T_β is the topology generated by D_β, is a complete metrizable locally convex space.

<u>Proof</u>: Let us first of all observe that the function $||a,\beta+\varepsilon||$ as defined in (4.3.2) is finite for each $\varepsilon > 0$ and $a \in \Gamma_\beta(p,q)$. Indeed, for $\varepsilon > 0$, choose $\delta > 0$ with $\delta < \varepsilon$. Then for $a \in \Gamma_\beta(p,q)$, there exists $N_0 \equiv N_0(\delta)$ $N_0 \equiv N_0(\delta)$ such that

$$(m+n)|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \beta+\delta, \quad \forall\, m+n \geq N_0$$

If $N_* \in \mathbb{N}$ is such that the inequalities

$$p_m^{-1} > h + p_{m-1}^{-1} \;;\; q_n^{-1} > k + q_{n-1}^{-1}, \quad \forall\, m,n \geq N_*$$

hold, then following the analysis as in the proof of Theorem 4.2.3, we can establish

$$\sum_{m+n \geq N^*} \sum |a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} \leq \sum_{m+n \geq N^*} \sum \left(\frac{\beta+\delta}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \infty$$

where $N^* \geq \max\{N_o, N_*\}$.

The expression in (4.3.2) is clearly a norm on $\Gamma_\beta(p,q)$ for each $\varepsilon > 0$ and also for $\varepsilon_1 > \varepsilon_2$, $||a,\beta+\varepsilon_1|| < ||a,\beta+\varepsilon_2||$.

If $\{r_n\}$ is a decreasing sequence of real numbers such that $r_n \to 0$, then the countable family $\{||.,\beta+r_n|| : n \geq 1\}$ also generates the topology T_β, thereby making $(\Gamma_\beta(p,q), T_\beta)$ a metrizable space.

For proving the completeness of the space $\Gamma_\beta(p,q)$, consider a Cauchy sequence $\{a^t\}$ in $\Gamma_\beta(p,q)$. So for each $\eta > 0$, there exists $t_o \equiv t_o(\eta)$ such that

$$||a^t-a^s,\beta+\varepsilon|| < \eta\,, \quad \forall\, t,s \geq t_o$$

$$(*) \quad \Longrightarrow |a^t_{oo}-a^s_{oo}| + \sum_{m+n \geq 1} \sum |a^t_{mn}-a^s_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta, \quad \forall\, t,s \geq t_o,$$

Therefore, for all $t, s \geq t_o$, we have

$$(+) \quad |a^t_{oo} - a^s_{oo}| < \eta; \quad |a^t_{mn} - a^s_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta, \quad \forall\, m, n, m+n \geq 1$$

Consequently, for each fixed pair (m,n) of positive integers, $\{a^t_{mn}\}$ is a Cauchy sequence in $\mathbb{K}$. Hence we can find a matrix $a = [a_{mn}]$ in Ω such that $a^t_{mn} \to a_{mn}$ as $t \to \infty$. Therefore, we have from (+),

$$|a^{t_o}_{oo} - a_{oo}| < \eta; \quad |a^{t_o}_{mn} - a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta, \quad \forall\, m+n \geq 1.$$

Since $a^{t_o} \in \Gamma_{\beta}(p,q)$, for $\varepsilon_o < \varepsilon$ and sufficiently large $m+n$, $m+n \geq N_o$ (say) we get

$$\begin{aligned}
\left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a_{mn}| &\leq \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a^{t_o}_{mn} - a_{mn}| + \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a^{t_o}_{mn}| \\
&< \eta + \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a^{t_o}_{mn}| \\
&< \eta + \left(\frac{\beta+\varepsilon_o}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}.
\end{aligned}$$

As $\left(\frac{\beta+\varepsilon_o}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} \to 0$ as $m+n \to \infty$, there exists a constant $L > 0$ such that

$$\left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a_{mn}| \leq L, \ \forall\, m,n \geq 1.$$

Hence

$$(m+n)\,|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq (\beta+\varepsilon)\, L^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$\Longrightarrow \overline{\lim}\,(m+n)\ |a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \beta.$$

This shows that $a \in \Gamma_\beta(p,q)$. Also, from (*), we have

$$|a_{oo}^t - a_{oo}^s| + \sum_{1\leq m+n\leq r}\sum |a_{mn}^t - a_{mn}^s| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta,$$

for $t,s \geq t_o$ and any $r \in \mathbb{N}$. Letting $s \to \infty$, we get

$$|a_{oo}^t - a_{oo}| + \sum_{1\leq m+n\leq r}\sum |a_{mn}^t - a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} \leq \eta,\ t \geq t_o.$$

As $r \in \mathbb{N}$ is arbitrary,

$$||a^t - a,\ \beta+\varepsilon|| \leq \eta\ ,\ \forall\, t \geq t_o\ .$$

Hence $a^t \to a$ in T_β. This completes the proof.

To characterize the dual of $\Gamma_\beta(p,q)$, we need the following

<u>Lemma</u> 4.3.4: For each $a \in \Gamma_\beta(p,q)$, the series

$$(+) \qquad \sum_{m+n>0}\sum (m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} a_{mn}c_{mn} ,$$

converges if and only if for some $\varepsilon > 0$,

$$(*) \qquad |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \frac{1}{\beta+\varepsilon} , \ \forall\, m,n \geq 1 \text{ and } |c_{oo}| \leq 1.$$

Proof: Suppose (*) is true. Choose $\delta > 0$ with $\delta < \varepsilon$. Then there exists an integer $N_o \equiv N_o(\delta)$ such that

$$(m+n) \ |a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \beta+\delta, \ \forall\, m+n \geq N_o.$$

Consequently, we have

$$\sum_{m+n \geq N_o}\sum \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a_{mn}c_{mn}| \leq \sum_{m+n \geq N_o}\sum \left(\frac{\beta+\delta}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}$$

$$< \infty$$

as $\beta+\delta < \beta+\varepsilon$ and $\{p_m^{-1}\},\{q_n^{-1}\}$ satisfy (4.3.1) (cf. proof of Theorem 4.2.3].

Conversely, suppose (+) converges, but (*) does not hold. So for each $k \geq 1$, there exist m_k and n_k such that

$$|c_{m_k n_k}| > \left(\frac{1}{\beta+1/k}\right)^{\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}} .$$

Define $a \in \Omega$ as follows:

$$(m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} a_{mn} = \begin{cases} (\beta+\frac{1}{k})^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} & m=m_k,\ n=n_k,\ k \geq 1; \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly,

$$\overline{\lim}(m+n)\,|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \beta$$

and so $a \in \Gamma_\beta(p,q)$. But

$$|(m_k+n_k)^{\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}} a_{m_k n_k}\, c_{m_k n_k}| > 1,\ \forall\, k \geq 1.$$

This contradicts the fact that, (+) converges for each $a \in \Gamma_\beta(p,q)$. Hence (*) holds if (+) converges.

Using the above lemma, we are now in a position to prove the main result of this subsection, contained in

<u>Theorem</u> 4.3.5: A continuous linear functional F on $\Gamma_\beta(p,q)$ is precisely of the form

$$(*) \quad F(a) = a_{00}c_{00} + \sum_{m+n\geq 1}\sum (m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} a_{mn}\, c_{mn}$$

where for some $\varepsilon > 0$

$$(+) \quad |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \frac{1}{\beta+\varepsilon},\ \forall\, m,n \geq 1 \ \text{ and } \ |c_{00}| \leq 1.$$

<u>Proof</u>: If (+) holds, (*) is well defined by the Lemma 4.3.4. Clearly F, given by (*), is a linear functional on $\Gamma_\beta(p,q)$. Moreover, the continuty of F is a consequence of the following inequality namely,

$$|F(a)| \leq |a_{oo}| + \sum_{m+n\geq 1}\sum (m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a_{mn}c_{mn}| \quad .$$

$$\leq |a_{oo}| + \sum_{m+n\geq 1}\sum |a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}$$

$$= ||a,\beta+\varepsilon||$$

Conversely, let F be an arbitrary continuous linear functional on $\Gamma_\beta(p,q)$. Recalling the matrices (e^{mn}) from Chapter 1. Section 3, for $a \in \Gamma_\beta(p,q)$, consider the sth-plane section $a^{(s)}$ given by

$$a^{(s)} = \sum_{0\leq m+n\leq s}\sum a_{mn}e^{mn} \quad .$$

Then

$$||a-a^{(s)},\beta+\varepsilon|| \leq \sum_{m+n>s}\sum |a_{mn}| \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}$$

$$\to 0 \text{ as } s \to \infty \quad .$$

Therefore, by continuity of F,

$$F(a) = \lim_{s \to \infty} F(a^{(s)})$$

$$(++) \qquad = a_{00}F(e^{00}) + \lim_{s \to \infty} \sum_{1 \leq m+n \leq s} \sum a_{mn}F(e^{mn})$$

Also, by continuty of F, we have

$$|F(a)| \leq ||a, \beta+\varepsilon||$$

for all $a \in \Gamma_{\beta}(p, q)$ and some $\varepsilon > 0$. Therefore if $c \in \Omega$ is given by

$$c_{mn} = \begin{cases} \dfrac{F(e^{mn})}{(m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}} & , m+n \neq 0; \\ F(e^{00}) & , m=0, n=0, \end{cases}$$

then,

$$|c_{00}| = |F(e^{00})| \leq ||e^{00}, \beta+\varepsilon|| = 1$$

and

$$|c_{mn}| = \frac{|F(e^{mn})|}{(m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}} < \frac{||e^{mn}, \beta+\varepsilon||}{(m+n)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}}$$

But

$$||e^{mn}, \beta+\varepsilon|| = \left(\frac{m+n}{\beta+\varepsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} , \forall\, m+n \geq 1,$$

therefore,

$$|c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \frac{1}{\beta+\varepsilon}, \ \forall\, m+n \geq 1$$

Thus c satisfies (+) and from (++) F has the representation as given in (*).

<u>The Space</u> $\Gamma_o(p,q)$

Before we pass on to topologize the space $\Gamma_o(p,q)$, we first consider the relationship between the spaces $\Gamma_o(p,q)$ and $\Gamma_o(r,s)$, defined corresponding to two different pairs (p,q) and (r,s) of sequences $p \equiv \{p_m\}, q \equiv \{q_n\}$, $r \equiv \{r_m\}$ and $s \equiv \{s_n\}$ satisfying the condition (4.3.1). Indeed we have

<u>Proposition</u> 4.3.6: $\Gamma_o(p,q) \subset \Gamma_o(r,s)$ if and only if

$$(4.3.7) \qquad \underline{\lim}\ \frac{r_m}{p_m} > 0 \quad \text{and} \quad \underline{\lim}\ \frac{s_n}{q_n} > 0.$$

<u>Proof</u>: Suppose (4.3.7) holds. Then, for λ and μ with

$$\underline{\lim}\ \frac{r_m}{p_m} > \lambda > 0 \quad \text{and} \quad \underline{\lim}\ \frac{s_n}{q_n} > \mu > 0,$$

there exist integers $N_1 \equiv N_1(\lambda)$ and $N_2 \equiv N_2(\mu)$ such that

$$r_m > \lambda p_m \ ; \ s_n > \mu\, q_n$$

for $m \geq N_1$ and $n \geq N_2$ respectively. Take $a \in \Gamma_o(p,q)$. Then we can find an integer $N_o \in \mathbb{N}$ such that

$$(*) \qquad |(m+n)!\ a_{mn}| \leq 1, \ m+n \geq N_o.$$

Now for $\theta = \min\{\lambda,\mu\}$ and $m \geq N_1$, $n \geq N_2$

$$\frac{1}{r_m} + \frac{1}{s_n} \leq \frac{1}{\lambda p_m} + \frac{1}{\mu q_n} \leq \frac{p_m+q_n}{\theta p_m q_n},$$

$$\Longrightarrow \frac{r_m s_n}{r_m+s_n} = \frac{1}{r_m^{-1}+s_n^{-1}} \geq \frac{\theta p_m q_n}{p_m+q_n}.$$

Therefore, in view of (*), we get

$$|(m+n)!\, a_{mn}|^{\frac{r_m s_n}{r_m+s_n}} \leq \left[|(m+n)!\, a_{mn}|^{\frac{p_m q_n}{p_m+q_n}}\right]^{\theta},$$

for $m \geq N_1$, $n \geq N_2$ and $m+n \geq N_0$. Thus $a \in \Gamma_0(r,s)$ and hence $\Gamma_0(p,q) \subset \Gamma_0(r,s)$.

Conversely, suppose (4.3.7) is not true. Consequently,

$$\underline{\lim}\, \frac{r_m}{p_m} = 0, \quad \underline{\lim}\, \frac{s_n}{q_n} = 0.$$

Therefore, there exist sequences $\{m_i\}$ and $\{n_j\}$ of positive integers such that

$$r_{m_i} < \frac{1}{i}\, p_{m_i} \; ; \; s_{n_j} < \frac{1}{j}\, q_{n_j}, \quad i \geq 1, \; j \geq 1.$$

Now, construct $a \in \Omega$ as follows

$$(m+n)!\, a_{mn} = \begin{cases} [(i+j)^{-1}]^{p_m^{-1}+q_n^{-1}}, & m=m_i,\; n=n_j;\, i,j \geq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly, a $\in \Gamma_o(p,q) \subset \Gamma_o(r,s)$. But

$$|(m_i+n_j)!\,a_{m_i n_j}|^{\frac{r_{m_i} s_{n_j}}{r_{m_i}+s_{n_j}}} = [(i+j)^{-1}]^{\frac{p_{m_i}^{-1}+q_{n_j}^{-1}}{r_{m_i}^{-1}+s_{n_j}^{-1}}}$$

$$\geq [(i+j)^{-1}]^{i^{-1}+j^{-1}}$$

yields

$$\{|(m_i+n_j)!\,a_{m_i n_j}|^{\frac{1}{r_{m_i}^{-1}+s_{n_j}^{-1}}}\} \not\to 0 \quad \text{as} \quad i+j \to \infty$$

This is a contradiction to the fact that $a \in \Gamma_o(r,s)$. Hence the required result follows.

For topologizing the space $\Gamma_o(p,q)$, we have two methods to generate the same locally convex metrizable topology T_o. To begin with, we introduce a real valued function Ψ on $\Gamma_o(p,q)$ as follows:

$$\Psi(a) = \sup\{|a_{oo}|;\ |(m+n)!\,a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}},\ m+n \neq 0\},$$

$$a \in \Gamma_o(p,q)$$

where

$$M = \max\{1, \sup_m p_m\};\ N = \max\{1, \sup_n q_n\}.$$

Then we have

Proposition 4.3.8: The function Ψ is a paranorm on the space $\Gamma_o(p,q)$ and $\Gamma_o(p,q)$ is a complete metric space where the metric d is being induced by the paranorm Ψ.

<u>Proof</u>: Clearly, $\Psi(a) \geq 0$ and $\Psi(a) = \Psi(-a)$ for all a in $\Gamma_o(p,q)$.

For $a = [a_{mn}]$ and $b = [b_{mn}]$ in $\Gamma_o(p,q)$ the inequality,

$$(*) \qquad |a_{mn}+b_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq |a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} + |b_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$\implies \Psi(a+b) \leq \Psi(a) + \Psi(b).$$

Now, let $\Psi(a^t-a) \to 0$ and $\lambda_t \to \lambda$ for $a^t, a \in \Gamma_o(p,q)$ and $\lambda_t, \lambda \in \mathbb{K}$. Then by (*)

$$(+) \qquad \Psi(\lambda_t a^t - \lambda a) \leq \Psi(\lambda_t(a^t-a)) + \Psi(a(\lambda_t-\lambda)).$$

For given $\varepsilon > 0$, we can find an integer $t_o \in \mathbb{N}$ such that

$$(*+) \qquad |\lambda_t - \lambda| < 1 \quad , \forall\, t \geq t_o$$

and

$$(**) \qquad \Psi(a^t-a) < \frac{\varepsilon}{2(1+|\lambda|)} \; , \; \forall\, t \geq t_o$$

Since $\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}} \leq 1$, for all $m,n \geq 0$, we have for all $m,n \geq 0$

$$|\lambda_t|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < (1+|\lambda|)^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq 1+|\lambda|, \; \forall\, t \geq t_o.$$

Consequently

$$(+*) \qquad \sup_{m,n} |\lambda_t|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < 1+|\lambda| \ , \ \forall\, t \geq t_o$$

Hence from $(**)$ and $(+*)$, for $t \geq t_o$ we have

$$\Psi(\lambda_t(a^t-a)) = \sup_{m,n} |(m+n)!\,(a^t_{mn}-a_{mn})\lambda_t|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$\leq \Psi(a^t-a) \sup_{m,n} |\lambda_t|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$< \varepsilon/2$$

Further, there exists an integer $N_o \equiv N_o(\varepsilon)$ such that

$$|(m+n)!\,a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon/2 \ , \quad \forall\, m+n \geq N_o$$

since $a \in \Gamma_o(p,q)$. Also from $(*+)$, it follows that

$$\sup_{m,n} |\lambda_t-\lambda|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq 1, \ \forall\, t \geq t_o$$

Hence

$$(++) \qquad \sup_{m+n \geq N_o} |(m+n)!\,a_{mn}(\lambda_t-\lambda)|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon/2, \ \forall\, t \geq t_o$$

Since

$$\sup_{0 \leq m+n < N_o} |\lambda_t-\lambda|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \to 0 \ \text{ as } \ t \to \infty \ ;$$

there exists $t_1 \equiv t_1(\varepsilon)$ so that

$$\sup_{0 \le m+n < N_o} |\lambda_t - \lambda|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \frac{\varepsilon}{2\Psi(a)} \quad , \forall\ t \ge t_1$$

Therefore, for $t \ge t_1$

$$(+++) \qquad \sup_{0 \le m+n < N_o} |(m+n)!\, a_{mn}(\lambda_t - \lambda)|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon/2$$

Thus, if $t \ge \max\{t_o, t_1\}$, we have

$$\sup_{m,n} |(m+n)!\, a_{mn}(\lambda_t - \lambda)|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon/2$$

Equivalently

$$\Psi(a(\lambda_t - \lambda)) < \varepsilon/2$$

Hence from (+), finally for $t \ge \max\{t_o, t_1\}$, we get

$$\Psi(\lambda_t a^t - \lambda a) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

This establishes that, Ψ is a paranorm on the space $\Gamma_o(p,q)$.

For proving completeness of $\Gamma_o(p,q)$, consider a Cauchy sequence $\{a^t\}$ in $\Gamma_o(p,q)$. So for given $\varepsilon > 0$, there exists $t^* \equiv t^*(\varepsilon)$ such that for $t, s \ge t^*$,

$$(***) \qquad |(m+n)!(a_{mn}^{t} - a_{mn}^{s})|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon^{M^{-1}+N^{-1}}, \ m,n \geq 0.$$

Consequently, the sequence $\{a_{mn}^{t}\}$ is Cauchy in the scalar field $\mathbb{K}$, for each fixed pair of (m,n). Hence we can find a set $\{a_{mn}, m,n \geq 0\} \subset \mathbb{K}$ such that

$$a_{mn}^{t} \to a_{mn} \quad \text{as} \quad t \to \infty$$

Also,

$$|(m+n)!a_{mn}|^{\frac{1}{p_m^{-1}+q_n^{-1}}} \leq |(m+n)!(a_{mn}^{t^*}-a_{mn})|^{\frac{1}{p_m^{-1}+q_n^{-1}}} + |(m+n)!a_{mn}^{t^*}|^{\frac{1}{p_m^{-1}+q_n^{-1}}}$$

$$< \varepsilon + |(m+n)!a_{mn}^{t^*}|^{\frac{1}{p_m^{-1}+q_n^{-1}}}$$

$\implies a \in \Gamma_0(p,q)$

Further from (***), taking the limit as $s \to \infty$, we get

$$\sup_{m,n} |(m+n)!(a_{mn}^{t}-a_{mn})|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon^{M^{-1}+N^{-1}}, \ \forall\ t \geq t^*$$

$\implies \Psi(a^{t}-a) \to 0$ as $t \to \infty$

Hence the space $\Gamma_0(p,q)$ is complete.

Next, we consider a family of seminorms generating a topology T_0 equivalent to the paranorm topology T_Ψ. For $a \in \Gamma_0(p,q)$ and $\delta > 0$, we introduce

$$||a,\delta|| = \sup_{m,n} \{|(m+n)!\, a_{mn}| / \delta^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}\}.$$

Let us observe the following

Proposition 4.3.9: The family $\{||\cdot,\delta||, \delta > 0\}$ is a decreasing family of seminorms on $\Gamma_o(p,q)$, generating a locally convex metrizable topology T_o.

Proof: Clearly, $||\cdot,\delta||$ is a seminorm on $\Gamma_o(p,q)$, for each $\delta > 0$; and for $\delta_1 \geq \delta_2$, $||a,\delta_1|| \leq ||a,\delta_2||$, for all $a \in \Gamma_o(p,q)$.

Also, for any sequence $\{r_n\}$ of positive real numbers with $r_n \to 0$, T_o is generated by the countable family $||\cdot,r_n||$ and so it is metrizable.

The topologies T_Ψ and T_o on $\Gamma_o(p,q)$ are essentially the same, as shown in

Proposition 4.3.10: The two topologies T_Ψ and T_o on the space $\Gamma_o(p,q)$ are equivalent.

Proof: Let $a^i \to 0$ in T_Ψ, $\delta > 0$ and $0 < \varepsilon < 1$ be given Choose $\lambda > 0$ such that $\lambda < \delta\varepsilon$, then we have

$$\left(\frac{\lambda}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < (\varepsilon)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \varepsilon, \ \forall\, m+n \geq 0$$

Since $a^i \to 0$ in T_Ψ, for the above choice of λ, there exists an integer $I \equiv I(\lambda)$ such that

$$|(m+n)!\,a^i_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \lambda, \ \forall\ m+n \geq 0,\ i \geq I.$$

Hence for $i \geq I$,

$$\frac{|(m+n)!\,a^i_{mn}|}{\delta^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}} < \left(\frac{\lambda}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \varepsilon, \ \forall\ m+n \geq 0$$

So $||a^i;\delta|| < \varepsilon$, $i \geq I$, that is to say, $a^i \to 0$ in T_o.

Conversely, let $a^i \to 0$ in T_o. So we get $||a^i,\delta|| \to 0$ for each $\delta > 0$. Then for each $\varepsilon > 0$, choose $\lambda, \delta > 0$ such that $\delta < \varepsilon$ and $\lambda < \frac{\varepsilon}{\delta}$. Then

$$\lambda < \left(\frac{\varepsilon}{\delta}\right) < \left(\frac{\varepsilon}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, \ \forall\ m,n \geq 0$$

$$\Longrightarrow \delta\,\lambda^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon, \quad \forall\ m,n \geq 0.$$

Now for the above choice of λ, there exists $I \equiv I(\lambda)$ such that for $i \geq I$

$$\frac{|(m+n)!\,a^i_{mn}|}{\delta^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}} < \lambda, \ \forall\ m,n \geq 0$$

$$\Longrightarrow |(m+n)!\,a^i_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \delta\,\lambda^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon,\ i \geq I,\ \forall\ m,n \geq 0$$

and hence the two topologies T_Ψ and T_o are equivalent.

To characterize the dual of $\Gamma_o(p,q)$, we need

Lemma 4.3.11: For every $a \in \Gamma_o(p,q)$, the series

$$(+) \qquad \sum\sum_{m+n \geq 0} (m+n)!\,a_{mn}c_{mn},$$

converges if and only if

$$(*) \qquad \{|c_{oo}|;\ |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}},\ m+n \geq 1\}$$

is bounded.

Proof: Assume first of all that there exists a positive constant L, such that

$$|c_{oo}| \leq L\ ;\ |c_{mn}| \leq L^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}},\ m+n \geq 1$$

For each $a \in \Gamma_o(p,q)$, we can find $\ell \in \mathbb{N}$ such that

$$|(m+n)!\,a_{mn}| \leq \left(\frac{1}{2L}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}},\ \forall\ m+n \geq \ell$$

Hence for $m+n \geq \ell$, we have

$$|(m+n)!\,a_{mn}c_{mn}| \leq 2^{-\left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)}$$

and this leads to the inequality

$$(**) \quad \leq \sum_{0 \leq m+n \leq \ell-1}\sum |(m+n)!\, a_{mn} c_{mn}| + \sum_{m+n \geq \ell}\sum 2^{-\left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)} .$$

Using Mandelbrojt restriction on p_m^{-1} and q_n^{-1} and following the proof of Theorem 4.2.3, we can establish

$$\sum_{m \geq 0} 2^{-\left(\frac{p_m^{-1}}{M^{-1}+N^{-1}}\right)} < \infty$$

and

$$\sum_{n \geq 0} 2^{-\left(\frac{q_n^{-1}}{M^{-1}+N^{-1}}\right)} < \infty$$

Hence from (**), the series (+) converges.

Conversely, assume that the series (+) is convergent for each $a \in \Gamma_0^*(p,q)$, but the sequence (*) is unbounded. Then there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ of integers such that

$$|c_{m_k n_k}| \geq k^{\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}} , \quad k \geq 1.$$

Now, construct $a \in \Omega$ as follows.

$$(m+n)!\, a_{mn} = \begin{cases} k^{-\left(\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}\right)} , & m=m_k;\ n=n_k\ ;\ k \geq 1; \\ 0 , & \text{otherwise} \end{cases}$$

Then

$$|(m+n)!\,a_{mn}|^{\frac{1}{p_m^{-1}+q_n^{-1}}} \to 0 \text{ as } m+n \to \infty$$

and so $a \in \Gamma_0(p,q)$. However,

$$|(m_k+n_k)!\,a_{m_k n_k}| \geq 1$$

and this contradicts the convergence of (+). Thus the required result follows.

The main result of this subsection is contained in

<u>Theorem</u> 4.3.12: Consider $\Gamma_0(p,q)$ equipped with either of the two topologies T_0 or T_Ψ. Then every continuous linear functional F on $\Gamma_0(p,q)$ is given by

$$(*) \qquad F(x) = \sum_{m+n \geq 0}\sum (m+n)!\,x_{mn}c_{mn}, \quad x \in \Gamma_0(p,q)$$

where $\{c_{mn}\}$ is such that

$$(+) \qquad \{|c_{00}|;\ |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}},\ m+n \neq 0\}$$

is bounded.

Conversely, for $c \in \Omega$ satisfying (+), the function defined by ($*$) is continuous and linear.

<u>Proof</u>. Suppose $F \in [\Gamma_0(p,q)]^*$. For $x \in \Gamma_0(p,q)$, observe that

$$\Psi(x-x^{(s)}) = \sup_{m+n>s} |(m+n)!\,x_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \to 0 \text{ as } s \to \infty,$$

where the s-th plane section $x^{(s)}$ of x is given by

$$x^{(s)} = \sum_{0\le m+n\le s}\sum x_{mn} e^{mn} .$$

Hence by the continuity of F we get $F(x^{(s)}) \to F(x)$, and so for $x \in \Gamma_0(p,q)$

$$F(x) = \lim_{s\to\infty} \sum_{0\le m+n\le s}\sum x_{mn} F(e^{mn})$$

$$= \sum_{m+n\ge 0}\sum (m+n)!\, x_{mn} c_{mn}$$

where

$$c_{mn} = \frac{1}{(m+n)!} F(e^{mn}).$$

Now by the Lemma 4.3.11, the sequence

$$\left\{ |c_{oo}| ; |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} , m+n \ne 0 \right\},$$

is bounded.

Conversely, let F be mentioned as above in (*) . We need show only the continuity of F, linearity being clear from the definition. So let $x^t \to 0$ in either of the two topologies. Put

$$L = \sup \left\{ |c_{oo}| ; |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} , m+n \ne 0 \right\}.$$

By Mandelbrojt restriction on p_m^{-1} and q_n^{-1} we can find an integer N_o such that

$$p_m^{-1}-p_{m-1}^{-1} \geq h;\ q_m^{-1}-q_{m-1}^{-1} \geq k,\ \forall\, m \geq N_o.$$

Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so small that

$\eta L < 1$ and

$$\eta L + \frac{(\eta L)^{\frac{p_o^{-1}+q_o^{-1}}{M^{-1}+N^{-1}}}}{[1-(\eta L)^{h/M^{-1}+N^{-1}}][1-(\eta L)^{k/M^{-1}+N^{-1}}]}[N_o(1-(\eta L)^{h/M^{-1}+N^{-1}})+1]$$

$$[N_o(1-(\eta L)^{k/M^{-1}+N^{-1}}) + 1] < \varepsilon$$

Then there exists $t_o \equiv t_o(\eta)$ such that, for $t \geq t_o$ we get

$$|x_{oo}^t| < \eta;\quad |(m+n)!x_{mn}^t|^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta,\ m+n \geq 1$$

But we have

$$|\Psi(x^t)| \leq |c_{oo}x_{oo}^t| + \sum_{m+n\geq 1}\sum |(m+n)!\,x_{mn}^t c_{mn}|$$

$$(**)\qquad \leq \eta L + \sum_{m+n\geq 1}\sum (\eta L)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}},\quad t \geq t_o$$

where

$$\sum_{m\geq 0} (\eta L)^{\frac{p_m^{-1}}{M^{-1}+N^{-1}}} = \sum_{m=0}^{N_o-1} (\eta L)^{\frac{p_m^{-1}}{M^{-1}+N^{-1}}} + \sum_{m\geq N_o} (\eta L)^{\frac{p_m}{M^{-1}+N^{-1}}}$$

$$p^{-1}$$

$$\leq \sum_{m=0}^{N_o-1} (\eta L)^{\frac{p_m^{-1}}{M^{-1}+N^{-1}}} + (\eta L)^{\frac{N_o}{M^{-1}+N^{-1}}}[1+(\eta L)^{\frac{h}{M^{-1}+N^{-1}}}+((\eta L)^{\frac{h}{M^{-1}+N^{-1}}})^2+\cdots$$

$$= \sum_{m=0}^{N_o-1} (\eta L)^{\frac{p_m^{-1}}{M^{-1}+N^{-1}}} + \frac{(\eta L)^{\frac{p_{N_o}^{-1}}{M^{-1}+N^{-1}}}}{1-(\eta L)^{h/M^{-1}+N^{-1}}}$$

$$\leq N_o (\eta L)^{\frac{p_o^{-1}}{M^{-1}+N^{-1}}} + \frac{(\eta L)^{\frac{p_o^{-1}}{M^{-1}+N^{-1}}}}{1-(\eta L)^{h/M^{-1}+N^{-1}}}$$

$$= \frac{(\eta L)^{\frac{p_o^{-1}}{M^{-1}+N^{-1}}}}{[1-(\eta L)^{h/M^{-1}+N^{-1}}]} [N_o (1-(\eta L)^{h/M^{-1}+N^{-1}}) + 1]$$

and similarly

$$\sum_{n\geq 0} (\eta L)^{\frac{q_n^{-1}}{M^{-1}+N^{-1}}} \leq \frac{(\eta L)^{\frac{q_o^{-1}}{M^{-1}+N^{-1}}}}{[1-(\eta L)^{k/M^{-1}+N^{-1}}]} [N_o (1-(\eta L)^{k/M^{-1}+N^{-1}}) + 1]$$

Hence from (**), for $t \geq t_o$

$$|\Psi(x^t)| \leq \eta L + \frac{(\eta L)^{\frac{p_o^{-1}+q_o^{-1}}{M^{-1}+N^{-1}}} [N_o (1-(\eta L)^{h/M^{-1}+N^{-1}})+1][N_o (1-(\eta L)^{\frac{k}{M^{-1}+N^{-1}}})+1]}{[1-(\eta L)^{h/M^{-1}+N^{-1}}][1-(\eta L)^{k/M^{-1}+N^{-1}}]}$$

$$< \varepsilon$$

Therefore $\Psi(x^t) \to 0$ as $t \to \infty$ and hence it is continuous.

For the final result of this chapter, we need

<u>Definition</u> 4.3.13: The sequences $p \equiv \{p_m, m \geq 0\}$ and $q \equiv \{q_n, n \geq 0\}$ are said to satisfy <u>density condition</u>

(4.3.14) $\qquad \underline{\lim}\, mp_m > M;\ \underline{\lim}\, nq_n > N$

where M and N have the usual meaning.

Restricting p and q further, we have the following relationship between the spaces $\Gamma_o(p,q)$ and $\Gamma_\beta(p,q)$:

Proposition 4.3.15: If p, q also satisfy the density condition (4.3.14), then $\Gamma_o(p,q)$ is a subspace of $\Gamma_\beta(p,q)$ for each $\beta > 0$.

Proof. Let $a \in \Gamma_o(p,q)$. Then

$$|(m+n)!\,a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \to 0 \text{ as } m+n \to \infty \quad .$$

So for each $\varepsilon > 0$, there exists an integer $N_o \equiv N_o(\varepsilon)$ such that

$$|(m+n)!\,a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon,\ \forall\, m+n \geq N_o$$

$$(+) \qquad \Longrightarrow (m+n)\,|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \frac{(m+n)\varepsilon}{[(m+n)!]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}},\ m+n \geq N_o$$

In order to show the boundedness of the right hand side sequence for $m+n \geq N_o$, let us observe

$$\frac{m+n}{\left[(m+n)!\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}} \cong c \, \frac{m+n}{\left[e^{-(m+n)}(m+n)^{m+n+1/2}\sqrt{2\pi}\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}}$$

where c is a constant (cf.[254] p. 131). Now from (4.3.1), there exists an integer $N_1 \in \mathbb{N}$ such that

$$p_m^{-1} > (m-N_1)h + p_{N_1}^{-1};\ q_n^{-1} > (n-N_1)k + q_{N_1}^{-1}\ ,\ \forall\ m,n \geq N_1$$

$$\Longrightarrow\ p_m^{-1} > (m-N_1)h\ ;\ q_n^{-1} > (n-N_1)k\ ,\ \forall\ m,n \geq N_1$$

$$\Longrightarrow\ p_m^{-1}+q_n^{-1} > (m+n)\ell - 2N_1\theta\ ,\qquad ,\ \forall\ m,n \geq N_1$$

where $\ell = \min\{h,k\}$, $\theta = \max\{h,k\}$. Therefore

$$e^{(m+n)\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < e^{\frac{M^{-1}+N^{-1}}{\ell-\frac{2N_1\theta}{m+n}}}$$

$$\to\ e^{\frac{M^{-1}+N^{-1}}{\ell}} \qquad \text{as } m+n \to \infty\ .$$

Hence there exists a constant K such that

$$(*)\qquad e^{(m+n)\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq K,\ \forall\ m+n \geq 0$$

Also, we can find $N_2 \in \mathbb{N}$ such that

$$mp_m M^{-1} \geq 1,\ nq_n N^{-1} \leq 1,\ \forall\ m,n \geq N_2$$

$$\Longrightarrow\quad (m+n)(M^{-1}+N^{-1}) \geq p_m^{-1}+q_n^{-1}\ ,\ \forall\ m,n \geq N_2$$

Hence

$$(**)\ \frac{m+n}{\left[(m+n)^{m+n+1/2}\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}} = \frac{m+n}{\left[(m+n)^{m+n}\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}} \cdot \frac{1}{\left[(m+n)^{1/2}\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}}$$

$$\leq \frac{m+n}{m+n}\ \frac{1}{\left[(m+n)^{1/2}\right]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}}\ ,\ \forall\ m,n \geq N_2$$

$$\to 1 \text{ as } m+n \to \infty$$

Since

$$(\sqrt{2\pi})^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \to 1$$

as $m+n \to \infty$, we have from (+), (*) and (**) the following

$$\overline{\lim}\ (m+n)\,|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} = 0.$$

Thus $a \in \Gamma_\beta(p,q)$ for all $\beta > 0$.

Chapter 5

$\alpha\mu$-Duals and Topologies

5.1 Introduction

In this chapter, we introduce a new concept of a dual λ_{α}^{μ} of a sequence space λ, where μ is another suitable sequence space and α a given member of ω. This new notion of a dual envelops, in particular, the well known α-, β- and γ-duals studied in [2], [153], [208], [92] and [29]. Some applications of this class of duals will be presented in the next chapter wherein we construct a class of weighted holomorphic functions with the help of these duals. In what follows, we essentially develop the basic results depending on these duals; in particular, we establish the results relating the sequential completeness of a sequence space λ with its $\alpha\mu$-perfectness, characterize boundedness of subsets of λ and so on so forth.

5.2 Basic Definitions and Results

In this section we introduce the notion of $\alpha\mu$-dual and define the $\alpha\mu$-topology. Indeed, throughout this chapter we fix the symbols α, μ, T_{μ} and D_{μ} to represent respectively a given sequence in ω, a sequence space,

a locally convex topology on μ and the family of semi-norms generating the topology T_μ. Then we define

<u>Definition</u> 5.2.1: The $\alpha\mu$-<u>dual</u> of a sequence space λ is the subset λ_α^μ of ω defined by

$$\lambda_\alpha^\mu = \{b \in \omega : \alpha ab \in \mu, \ \forall a \in \lambda\}.$$

Clearly, λ_α^μ is a subspace of ω. Similarly, we can introduce another subspace, namely, $\alpha\mu$-dual $\lambda_{\alpha\alpha}^{\mu\mu}$ of λ_α^μ, of ω, where

$$\lambda_{\alpha\alpha}^{\mu\mu} = (\lambda_\alpha^\mu)_\alpha^\mu = \{c \in \omega : \alpha bc \in \mu, \ \forall b \in \lambda_\alpha^\mu\}.$$

It is obvious that $\lambda \subset \lambda_{\alpha\alpha}^{\mu\mu}$. However, in case of equality, we have

<u>Definition</u> 5.2.2: A sequence space λ is said to be $\alpha\mu$-<u>perfect</u> if $\lambda = \lambda_{\alpha\alpha}^{\mu\mu}$.

To topologize either of the spaces λ and λ_α^μ, for $b \in \lambda_\alpha^\mu$ and $p \in D_\mu$, we define

$$(5.2.3) \qquad p_b^\alpha(a) = p(\{\alpha_n a_n b_n\}), \ a \in \lambda.$$

It is obvious that each p_b^α, $b \in \lambda_\alpha^\mu$ defines a seminorm on λ. Then the topology generated by the family $\{p_b^\alpha : p \in D_\mu, \ b \in \lambda_\alpha^\mu\}$ of seminorms, is called the $\alpha\mu$-<u>topology</u> and is denoted by $T_{\alpha\mu}$.

Similarly, for $a \in \lambda$ and $p \in D_\mu$, let us define

(5.2.4) $\qquad p_a^\alpha(b) = p(\{\alpha_n a_n b_n\}),\ b \in \lambda_\alpha^\mu.$

Clearly, each p_a^α, $a \in \lambda$ is a seminorm on λ_α^μ and the topology, denoted by $T_{\alpha\mu}^*$, is generated by the family $\{p_a^\alpha; a \in \lambda,\ p \in D_\mu\}$ of seminorms, is called $\alpha\mu$-<u>topology</u> on λ_α^μ.

<u>Note</u>: From now onwards, <u>we shall assume throughout</u> that $\alpha \in \omega$ is such that $\alpha_n \neq 0$, for each $n \geq 1$.

As particular cases of α and μ, let us observe the following:

(a) for $\alpha_n = 1$, $n \geq 1$, we have the following well known duals and perfectness;

(i) if $\mu = \ell^1$, λ_α^μ is the Köthe dual λ^x of λ and $\alpha\mu$-perfectness corresponds to Köthe perfectness;

(ii) if $\mu = cs$, λ_α^μ is the β-dual λ^β of λ and $\alpha\mu$-perfectness corresponds to β-perfectness;

(iii) if $\mu = bs$, λ_α^μ is the γ-dual λ^γ of λ and $\alpha\mu$-perfectness corresponds to γ-perfectness; and

(b) for $\alpha_n = \frac{1}{n}$, $n \geq 1$, $\mu = c_0$ and

$$\lambda = \{a \in \omega: \frac{a_n}{n} \to 0 \text{ as } n \to \infty\},$$

λ_α^μ is the space introduced by Boland (cf. [29], Definition 1.3, p. 41 and also chapter 2, section 2).

Thus, the $\alpha\mu$-dual as defined in Definition 5.2.1 yields various duals of λ for different values of α and different sequence spaces μ.

In this as well as in the next section, we shall study the impact of the structure of (μ, T_μ) on the spaces $(\lambda, T_{\alpha\mu})$ and $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$, and vice-versa, for an arbitrary α with $\alpha_n \neq 0$, $n \geq 1$. Let us begin with

Proposition 5.2.5: If (μ, T_μ) is a K-space (resp. an AK-space), then $(\lambda, T_{\alpha\mu})$ is also a K-space (resp. an AK-space). Similar result holds for the space $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$.

Proof: For showing the k-property of $(\lambda, T_{\alpha\mu})$, consider a net $\{a^\beta : \beta \in I\}$ in λ with the property that $a^\beta \to 0$ in $T_{\alpha\mu}$. Therefore, for $\varepsilon > 0$, $p \in D_\mu$ and $b \in \lambda_\alpha^\mu$, there exists an index $\beta_0 \equiv \beta_0\ (\varepsilon, p, b)$ in I such that

$$p_b^\alpha(a^\beta) = p(\{\alpha_n a_n^\beta b_n\}) < \varepsilon,\ \forall\ \beta \geq \beta_0.$$

Thus $\alpha a^\beta b \to 0$ in (μ, T_μ) for each $b \in \lambda_\alpha^\mu$.

Consequently,

$$\alpha_n a_n^\beta b_n \to 0,\ \forall\ n \geq 1,\ \{b_n\} \in \lambda_\alpha^\mu.$$

Taking $b = e^n$, we get

$$a_n^\beta \to 0,\ \forall\ n \geq 1.$$

Hence $(\lambda, T_{\alpha\mu})$ is a K-space.

For proving the AK-property of $(\lambda, T_{\alpha\mu})$, let us consider an element a in λ. Then for $p \in D_\mu$, $b \in \lambda_\alpha^\mu$ and $\gamma = \alpha ab \in \mu$, the equality

$$\begin{aligned} p_b^\alpha(a^{(n)}-a) &= p(\{\alpha_i(a_i^{(n)}-a_i)b_i\}) \\ &= p(\gamma^{(n)}-\gamma) \end{aligned}$$

along with the AK-ness of (μ, T_μ) yields the desired result.

The result for $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$ follows analogously.

5.3 $\alpha\mu$-Perfectness and Completeness

As mentioned earlier, we investigate in this section the relationship of $\alpha\mu$-perfectness of λ with the completeness or sequential completeness of (μ, T_μ). First of all, let us make the following simple observation contained in

Proposition 5.3.1 The $\alpha\mu$-dual λ_α^μ of a sequence space λ is always $\alpha\mu$-perfect.

Proof: Straightforward.

Next, we prove

Proposition 5.3.2: Let (μ, T_μ) be an AK-space. If $(\lambda, T_{\alpha\mu})$ is sequentially complete, then λ is $\alpha\mu$-perfect.

Proof: For proving the result, we just need show that $\lambda_{\alpha\alpha}^{\mu\mu} \subset \lambda$, for the other inclusion is always true. Let us, therefore, take an element c in $\lambda_{\alpha\alpha}^{\mu\mu}$. Then

$$c^{(n)} \in \lambda, \ \forall\, n \geq 1.$$

Also, for b in λ^{μ}_{α}, $\gamma^{(n)} \to \gamma$ in T_{μ}, where $\gamma = \alpha bc$. Therefore, for $p \in D_{\mu}$ and $m < n$, the equality

$$p^{\alpha}_{b}(c^{(n)} - c^{(m)}) = p(\gamma^{(n)} - \gamma^{(m)}),$$

yields that $\{c^{(n)}\}$ is a Cauchy sequence in $(\lambda, T_{\alpha\mu})$. Hence there exists an s in λ such that

$$c^{(n)} \to s \quad \text{as} \quad n \to \infty$$

relative to $T_{\alpha\mu}$. But $s=c$ by Proposition 5.2.5 and the fact that $c^{(n)}_{i} = c_{i}$, for each $n \geq i$. Hence $\lambda^{\mu\mu}_{\alpha\alpha} = \lambda$

Remark: Before we prove a partial converse of Proposition 5.3.2, let us note that none of the conditions, namely, AK-ness of the space (μ, T_{μ}) and the sequential completeness of $(\lambda, T_{\alpha\mu})$ is indispensible in the hypothesis of the above result; for we have

Example 5.3.3: Let μ be the non-AK-space ℓ^{∞} of all bounded sequence equipped with the usual supnorm topology T_{μ} and λ be the space c_o of all null sequences. Choose $\alpha = e$. Then

$$\begin{aligned} \lambda^{\mu}_{\alpha} = c_{o\,e}^{\ell^{\infty}} &= \{b \in \omega : ab \in \ell^{\infty}, \ \forall\, a \in c_o\} \\ &= \ell^{\infty} \end{aligned}$$

and

$$\lambda^{\mu\mu}_{\alpha\alpha} = (c_o^{\ell^\infty_e})^{\ell^\infty}_e = \{c \in \omega: bc \in \ell^\infty, \forall\, b \in \ell^\infty\}$$
$$= \ell^\infty$$

Thus λ is not $\alpha\mu$-perfect. Further observe that, $T_{e\ell^\infty}$ on c_o is generated by the family $\{p^e_b : b \in \ell^\infty\}$, of seminorms where

$$p^e_b(a) = \sup_n |a_n b_n|\ , \ a \in c_o,\ b \in \ell^\infty.$$

Since

$$\sup_n |a_n b_n| \leq ||a||_\infty ||b||_\infty$$

for each $a \in c_o$ and $b \in \ell^\infty$, $T_{e\ell^\infty}$ is weaker than T_μ. Clearly, $T_\mu \subset T_{e\ell^\infty}$, for $e \in \ell^\infty$. Hence, $T_{e\ell^\infty}$ on c_o is equivalent to the supnorm topology and so $(c_o, T_{e\ell^\infty})$ is complete.

<u>Example</u> 5.3.4: Let (μ, T_μ) be the AK-space φ, equipped with supnorm $||\cdot||_\infty$ and λ be c_o. For $\alpha = e$, we have

$$\lambda^\mu_\alpha = c_o^{\varphi_e} = \{b \in \omega: ab \in \varphi, \forall\, a \in c_o\}$$
$$= \varphi$$

and

$$\lambda^{\mu\mu}_{\alpha\alpha} = (c_o^{\varphi_e})^{\varphi}_e = \{c \in \omega: bc \in \varphi, \forall\, b \in \varphi\}$$
$$= \omega$$

Thus c_o is not $\alpha\mu$-perfect. The topology $T_{e\varphi}$ on λ is generated by the family $\{p_b^e : b \in \varphi\}$, where

$$p_b^e(a) = \sup_n |a_n b_n| \ , \ a \in c_o, \ b \in \varphi \ ,$$

since,

$$p_b^e(a) \leq \sum_n |a_n b_n| = p_b(a), \ \forall \ a \in c_o, \ b \in \varphi,$$

$T_{e\varphi} \subset \eta(c_o, \varphi)$. On the other hand, consider $b \in \varphi$ and ℓ be the length of b (cf. Definition 1.3.1). Then for $a \in c_o$,

$$p_b(a) = \sum_n |a_n b_n| \leq \ell \, p_b^e(a)$$

yields $\eta(c_o, \varphi) \subset T_{e\varphi}$. Consequently, $T_{e\varphi} = \eta(c_o, \varphi)$ and $(\lambda, T_{\alpha\mu}) \equiv (c_o, \eta(c_o, \varphi))$. However, the space $(c_o, \eta(c_o, \varphi))$ is not sequentially complete; for the sequence, $\{e^{(n)}, n \geq 1\}$ in c_o, where $e^{(n)} = \{1, \ldots, \underset{\text{nth-place}}{1}, \ldots\}$ is a nonconvergent $\eta(c_o, \varphi)$- Cauchy sequence in c_o.

On the contrary, the following example illustrates that the AK-ness of the space (μ, T_μ) is not a necessary condition in Proposition 5.3.2

Example 5.3.5: Let (μ, T_μ) be as in Example 5.3.3, and λ be φ. For $\alpha = e$

$$\lambda_\alpha^\mu = \varphi_e^{\ell^\infty} = \{b \in \omega : ab \in \ell^\infty, \ \forall \ a \in \varphi\}$$
$$= \omega$$

and

$$\lambda^{\mu\mu}_{\alpha\alpha} = (\varphi_e^{\ell^\infty})_e^{\ell^\infty} = \{c \in \omega : bc \in \ell^\infty, \forall b \in \omega\}$$
$$= \varphi$$

Thus λ is $\alpha\mu$-perfect. Observe that the topology $T_{e\ell^\infty}$ is generated by the family $\{p_b^e : b \in \ell^\infty\}$ of seminorms, where

$$p_b^e(a) = \sup_n |a_n b_n| , \; a \in \varphi , \; b \in \omega$$

Clearly, $T_{e\ell^\infty} \subset \eta(\varphi,\omega)$. For the other inclusion observe that, for each $b \in \omega$, $b_n \geq 0$, there exists $c \in \omega$, $c_n \geq 0$ such that $\{b_n/c_n\} \in \ell^1$ (cf. Proposition 1.3.8) Therefore, for $a \in \varphi$

$$p_b(a) = \sum_{n\geq 1} |a_n b_n|$$
$$\leq \Big(\sum_{n\geq 1} \frac{b_n}{c_n}\Big) p_c^e(a)$$

Consequently, $\eta(\varphi,\omega) \subset T_{e\ell^\infty}$. Hence $(\varphi, T_{e\ell^\infty}) \equiv (\varphi, \eta(\varphi,\omega))$ is complete by Proposition 1.3.2.

Converse of Proposition 5.3.2 is obtained in the form of

Proposition 5.3.6: Let (μ, T_μ) be a complete [resp. sequentially complete] K-space. If λ is $\alpha\mu$-perfect, then $(\lambda, T_{\alpha\mu})$ is complete (resp. sequentially complete)

Proof: Let us prove the result for completeness; the part for sequential completeness follows analogously.

Let $\{a^\beta : \beta \in I\}$ be a $T_{\alpha\mu}$-Cauchy net λ. Then by Proposition 5.2.5, it follows that $\{a_n^\beta\}$ is Cauchy in $\mathbb{K}$. Hence we get $a \in \omega$ such that

(*) $$a_n^\beta \to a_n, \ \forall\ n \geq 1$$

Now if $b \in \lambda_\alpha^\mu$, then for each $\beta \in I$, we have

$$\{\alpha_n a_n^\beta b_n\} \in \mu$$

So the relation

$$p_b^\alpha(a^\beta - a^\gamma) = p(\{\alpha_n a_n^\beta b_n\} - \{\alpha_n a_n^\gamma b_n\})$$

and the Cauchy character of $\{a^\beta : \beta \in I\}$ in $(\lambda, T_{\alpha\mu})$ show that $\{\alpha_n a_n^\beta b_n\}$ is a Cauchy net in μ. As (μ, T_μ) is complete, there exists $s \in \mu$ such that

(**) $$\{\alpha_n a_n^\beta b_n\} \underset{\beta}{\to} \{s_n\}$$

in the topology T_μ. Hence from (*) and (**) we get

$$s_n = \alpha_n a_n b_n \quad \forall\ n \geq 1$$

$$\Longrightarrow \{\alpha_n a_n b_n\} \in \mu \ .$$

As $b \in \lambda_\alpha^\mu$ is arbitrary, it follows that $a \in \lambda_{\alpha\alpha}^{\mu\mu} = \lambda$. Further the completeness of the space $(\lambda, T_{\alpha\mu})$ follows from the fact that, convergence of

$$\{\alpha_n a_n^\beta b_n\} \underset{\beta}{\to} \{s_n\}$$

in T_μ implies that $a^\beta \to a$ in $T_{\alpha\mu}$.

<u>Remark</u>: Neither the completeness of (μ, T_μ) nor the $\alpha\mu$-perfectness of λ can be dropped in the above proposition; for we have

<u>Example</u> 5.3.7: Let $\mu = \ell^1$ and T_μ be the supnorm topology on ℓ^1. Further, take $\lambda = \ell^1$ and $\alpha = e$. Then

$$\lambda_\alpha^\mu = \{b \in \omega : ab \in \ell^1, \ \forall\, a \in \ell^1\}$$
$$= \ell^\infty$$

and

$$(\lambda_\alpha^\mu)_\alpha^\mu = \{c \in \omega : bc \in \ell^1, \ \forall\, b \in \ell^\infty\}$$
$$= \ell^1$$

Thus λ is $\alpha\mu$-perfect. At the same time, we observe that the topology $T_{e\ell^1}$ is generated by the family $\{p_b^e : b \in \ell^\infty\}$, where

$$p_b^e(a) = \sup_n |a_n b_n| \ , \ a \in \ell^1, \ b \in \ell^\infty$$

Since $e \in \ell^\infty$, $T_\mu \subset T_{e\ell^1}$. For the other inclusion we have

$$p_b^e(a) = \sup_n |a_n b_n| \ , \ b \in \ell^\infty, \ a \in \ell^1$$
$$\leq c \sup_n |a_n| \ ,$$

where $c = \sup_n |b_n|$. Therefore $T_\mu = T_{e\ell^1}$. Hence $(\lambda, T_{\alpha\mu}) = (\ell^1 \|\cdot\|_\infty)$ is not complete

<u>Example</u> 5.3.8. This is the well known example of $\lambda = c_o$ and $(\mu, T_\mu) = (\ell^1, ||\cdot||_1)$, where $||\cdot||_1$ is the usual norm on ℓ^1. For $\alpha = e$,

$$\lambda_\alpha^\mu = \{b \in \omega : ab \in \ell^1, \forall a \in c_o\}$$
$$= \ell^1$$

and

$$(\lambda_\alpha^\mu)_\alpha^\mu = \{c \in \omega : bc \in \ell^1, \forall b \in \ell^1\}$$
$$= \ell^\infty$$

Thus c_o is not $\alpha\mu$-perfect. At the same time, $T_{e\ell^1}$ is generated by family $\{p_b^e : b \in \ell^1\}$, where

$$p_b^e(a) = \sum_{n\geq 1} |a_n b_n|, \quad b \in \ell^1, \; a \in c_o$$
$$= p_b(a) \qquad , \; b \in \ell^1$$

So the topology $T_{e\ell^1}$ on c_o is nothing but $\eta(c_o, \ell^1)$, and hence $(\lambda, T_{\alpha\mu}) = (c_o, \eta(c_o, \ell^1))$ is not complete by the Proposition 1.3.7.

On the other hand, completeness of (μ, T_μ) is not a necessary condition as illustrated in

<u>Example</u> 5.3.9: Consider the incomplete space $(\varphi, ||\cdot||_\infty)$ as (μ, T_μ) and $\lambda = \varphi$. For $\alpha = e$,

$$\lambda_\alpha^\mu = \varphi_e^\varphi = \{b \in \omega : ab \in \varphi, \forall a \in \varphi\}$$
$$= \omega$$

and

$$\lambda_{\alpha\alpha}^{\mu\mu} = (\varphi_e^{\varphi})_e^{\varphi} = \{c \in \omega : bc \in \varphi \ , \ \forall\, b \in \omega\}$$
$$= \varphi$$

Also, the space $(\varphi, T_{\alpha\mu})$ is complete since $T_{\alpha\mu} = \eta(\varphi, \omega)$, (cf. Example 5.3.5).

In view of Proposition 5.3.1, we derive from Proposition 5.3.6, the following

Corollary 5.3.10: If (μ, T_μ) is a complete (resp. sequentially complete) K-space, then $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$ is complete (resp. sequentially complete).

Combining Propositions 5.3.2 and 5.3.6, we get a characterization of $\alpha\mu$-perfectness exhibited in

Theorem 5.3.11: For a sequentially complete AK-space (μ, T_μ), λ is $\alpha\mu$-perfect if and only if $(\lambda, T_{\alpha\mu})$ is sequentially complete.

5.4 Boundedness

In this section we introduce the notion of complete boundedness in λ which is stronger than usual boundedness in $\alpha\mu$-topology and investigate conditions under which a bounded set is completely bounded.

First of all, we observe that a subset A of λ is $T_{\alpha\mu}$-bounded if and only if the set

$$Ab\alpha = \{\{a_i b_i \alpha_i\} : a \in A\}$$

is bounded in (μ, T_μ) for each $b \in \lambda_\alpha^\mu$. Replacing the singleton set $\{b\}$ in λ_α^μ by a $T_{\alpha\mu}^*$-bounded subset of λ_α^μ, we introduce

<u>Definition</u> 5.4.1: A subset A of λ is said to be <u>completely bounded</u> in λ if for each $T_{\alpha\mu}^*$-bounded subset B of λ_α^μ, the set

$$AB\alpha = \{\{a_i b_i \alpha_i\} : a \in A, b \in B\}$$

is bounded in (μ, T_μ).

<u>Remark</u>: Clearly, every completely bounded subset of λ is bounded relative to the topology $T_{\alpha\mu}$, for arbitrary α and μ.

Regarding converse of the above statement we prove

<u>Proposition</u> 5.4.2: Let (μ, T_μ) be a sequentially complete K-space. Then every $T_{\alpha\mu}$-bounded subset of λ is completely bounded.

<u>Proof</u>: Let us assume that the result is not true. Then there exists a $T_{\alpha\mu}$-bounded subset A of λ, which is not completely bounded. Hence we can find a $T_{\alpha\mu}^*$-bounded set B in λ_α^μ such that

$$AB\alpha = \{ab\alpha : a \in A, b \in B\}$$

is unbounded in (μ, T_μ). Consequently, for given $\varepsilon > 0$ and $p \in D_\mu$, there exists $a^1 \in A$, $b^1 \in B$ with the property that

$$p(a^1 b^1 \alpha) \geq 1+\varepsilon.$$

As A and B are bounded in $(\lambda, T_{\alpha\mu})$ and $(\lambda_\alpha^\mu, T_{\alpha\mu}^*)$ respectively, there exist constants $L_1 > 0$ and $M_1 > 0$ satisfying

$$\sup_{a \in A} p(\alpha a b^1) \leq L_1$$

and

$$\sup_{b \in B} p(\alpha a^1 b) \leq M_1.$$

Choose $m_1 \in \mathbb{N}$ such that

$$2^{-m_1+1} < \varepsilon / M_1.$$

From the unboundedness of $AB\alpha$, choose $a^2 \in A$ and $b^2 \in B$ such that

$$p(a^2 b^2 \alpha) \geq 2^{m_1}(L_1+2+\varepsilon).$$

Corresponding to the points $a^2 \in A$ and $b^2 \in B$, we can now find constants $L_2 > 0$ and $M_2 > 0$ such that

$$\sup_{a \in A} p(\alpha a b^2) \leq L_2$$

and

$$\sup_{b \in B} p(\alpha a^2 b) \leq M_2.$$

Choose $m_2 > m_1$ such that

$$2^{-m_2+1} < \varepsilon/M_2.$$

As above for the constant $2^{m_2}(L_1+2^{-m_1}L_2+3+\varepsilon)$, select $a^3 \in A$, $b^3 \in B$ and then the constants $L_3 > 0$, $M_3 > 0$ satisfying the relations

$$p(\alpha a^3 b^3) \geq 2^{m_2}(L_1+2^{-m_1}L_2+3+\varepsilon);$$

$$\sup_{a \in A} p(\alpha a b^3) \leq L_3;$$

and

$$\sup_{b \in B} p(\alpha a^3 b) \leq M_3.$$

Then consider $m_3 > m_2$ with the property

$$2^{-m_3+1} < \varepsilon/M_3.$$

Continuing this process, we get sequences $\{a^n\} \subset A$, $\{b^n\} \subset B$, constants $L_n > 0$, $M_n > 0$ and an increasing sequence $\{m_n\}$ of integers such that the following four inequalities hold:

$$p(\alpha a^n b^n) \geq 2^{m_{n-1}}\left(\sum_{i=0}^{n-1} 2^{-m_{i-1}} L_i+n+\varepsilon\right), \; L_0=0;$$

$$\sup_{a \in A} p(\alpha a b^n) \leq L_n;$$

$$\sup_{b \in B} p(\alpha a^n b) \leq M_n;$$

and

$$2^{-m_n+1} < \varepsilon / M_n$$

for $n=1,2,3,\ldots$

Now using Proposition 1.2.2, we infer that the sequence $\{\sum_{i=1}^{n} 2^{-m_{i-1}} b^i\}_n$ is a $T^*_{\alpha\mu}$-Cauchy sequence in λ^{μ}_{α}, since $\{2^{-m_{i-1}}\} \in \ell^1$ and $\{b^i\} \subset B$. So by Corollary 5.3.10. There exists $b^o \in \lambda^{\mu}_{\alpha}$ such that

$$b^o = T^*_{\alpha\mu}\text{-}\lim \sum_{i=1}^{n} 2^{-m_{i-1}} b^i = \sum_{i\geq 1} 2^{-m_{i-1}} b^i$$

Then for $n \geq 1$,

$$\begin{aligned}
p^{\alpha}_{b^o}(a^n) &= p(\alpha a^n \sum_{i\geq 1} 2^{-m_{i-1}} b^i) \\
&\geq 2^{-m_{n-1}} p(\alpha a^n b^n) - p(\alpha a^n b^1) \\
&\quad -2^{-m_1} p(\alpha a^n b^2) - \cdots - 2^{-m_{n-2}} p(\alpha a^n b^{n-1}) \\
&\quad -2^{-m_n} M_n (1 + 2^{m_n - m_{n+1}} + 2^{m_n - m_{n+2}} + \cdots) \\
&\geq (\sum_{i=0}^{n-1} 2^{-m_{i-1}} L_i + n + \varepsilon) - L_1 - 2^{-m_1} L_2 - \cdots \\
&\quad -2^{-m_{n-2}} L_{n-1} - \varepsilon \\
&= n
\end{aligned}$$

Hence A is $T_{\alpha\mu}$-unbounded. This contradiction prove the result.

Note Let us observe in the following example that the conclusion of Proposition 5.4.2 may hold even if (μ, T_μ) is not sequentially complete.

Example 5.4.3: Let α, λ and (μ, T_μ) be as in Example 5.3.9. Here $T_{\alpha\mu} = \eta(\varphi, \omega)$ and $T^*_{\alpha\mu} = \eta(\omega, \varphi)$. If A and B are respectively $\eta(\varphi,\omega)$ and $\eta(\omega,\varphi)$ bounded sets, then by Propositions 1.3.3 and 1.3.4, A is a set of bounded length say ℓ, and there exists sequences $\{r_i\}$ and $\{s_i\}$ in ω such that

$$|a_i| \leq r_i \,, \ \forall\, i \geq 1, \ a \in A$$

and

$$|b_i| \leq s_i \,, \ \forall\, i \geq 1, \ b \in B$$

Hence the set AB which has length ℓ, is bounded in (μ, T_μ).

Chapter 6

Holomorphic (Nuclear) Mappings

6.1 Introduction

In this chapter, we make use of our study on sequence spaces carried out in the preceding chapter. Indeed, corresponding to a sequence space μ, we define a class of weighted holomorphic functions on a Banach space, the weights having been provided by μ and introduce a subclass of this class related to the $\alpha\mu$-dual of a sequence space λ. This class of weighted holomorphic functions extend sufficiently a similar class studied earlier in [29]. After equipping this class of holomorphic functions with suitable locally convex topologies, we study the usual topological structure of these spaces. As we proceed further, we characterize bounded sets of this class and investigate conditions under which the subspace topology coincides with various other topologies on bounded sets. The last section is devoted to the characterization of relatively compact sets.

6.2 Certain Classess of Holomorphic (Nuclear) Mappings

In this section we study several subspaces of the class $H(E)$ of holomorphic mappings (cf. Section 5, Chapter 1) defined corresponding to an arbitrary normal sequence space μ, a nonzero α in ω and the $\alpha\mu$-dual of a sequence space λ. Indeed, we endow these spaces with suitable locally convex topologies in order to study their topological behaviour. Throughout this chapter, we consider sequences defined over $\mathbb{N}_0$ and the power $1/n$ whereever it appears for $n = 0$, means one.

Let us assume throughout that μ denotes a normal sequence space equipped with a Hausdorff locally convex topology T_μ generated by the family D_μ of solid seminorms, α a sequence in ω with $\alpha_n \neq 0$, $n \geq 0$ and λ a sequence space. Then we introduce the spaces

$$(6.2.1) \quad H^\mu(E) = \{f \in H(E): \hat{d}^n f(0) \in \mathcal{P}(^nE),\ n \geq 0 \text{ with } \{(\frac{\|\hat{d}^n f(0)\|}{n!})^{1/n}\} \in \mu\},$$

$$(6.2.2) \quad H_N^\mu(E) = \{f \in H(E): \hat{d}^n f(0) \in \mathcal{P}_N(^nE),\ n \geq 0 \text{ with } \{(\frac{\|\hat{d}^n f(0)\|_N}{n!})^{1/n}\} \in \mu\};$$

$$(6.2.3) \quad H_\alpha^\mu(E,\lambda) = \{f \in H^\mu(E): \{\|\hat{d}^n f(0)\|^{1/n}\} \in \lambda_\alpha^\mu\}$$

and

(6.2.4) $H^{\mu}_{N\alpha}(E,\lambda) = \{f \in H^{\mu}_{N}(E) : \{\| \hat{d}^n f(0)\|_N^{1/n}\} \in \lambda^{\mu}_{\alpha}\}$

Clearly

$$H^{\mu}_{N}(E,\lambda) \subset H^{\mu}(E)$$

and

$$H^{\mu}_{N\alpha}(E,\lambda) = H^{\mu}_{\alpha}(E,\lambda) \cap H^{\mu}_{N}(E)$$

On the spaces defined in (6.2.1), (6.2.2), (6.2.3) and (6.2.4), for $p \in D_{\mu}$ and $a \in \lambda$, let us define respectively the positive real valued functions Q_p, Q^N_p, $Q_{p,a}$ and $Q^N_{p,a}$ given by

(6.2.5) $Q_p(f) = p(\{(\frac{\|\hat{d}^n f(0)\|}{n!})^{1/n}\}),\ f \in H^{\mu}(E)$

(6.2.6) $Q^N_p(f) = p(\{(\frac{\|\hat{d}^n f(0)\|_N}{n!})^{1/n}\}),\ f \in H^{\mu}_N(E)$

(6.2.7) $Q_{p,a}(f) = p(\{\|\hat{d}^n f(0)\|^{1/n}\alpha_n a_n\}),\ f \in H^{\mu}_{\alpha}(E,\lambda)$

and

(6.2.8) $Q^N_{p,a}(f) = p(\{\|\hat{d}^n f(0)\|_N^{1/n}\ \alpha_n a_n\}),\ f \in H^{\mu}_{N\alpha}(E,\lambda).$

Then we have

<u>Proposition</u> 6.2.9: The collections $D_h = \{Q_p : p \in D_{\mu}\}$ $D^N_h = \{Q^N_p : p \in D_{\mu}\}$, $D^{\lambda}_h = \{Q_{p,a}\ p \in D_{\mu},\ a \in \lambda\}$ and $D^{N\lambda}_h = \{Q^N_{p,a} : p \in D_{\mu}, a \in \lambda\}$ define a separating family of

seminorms on $H^{\mu}(E)$, $H_N^{\mu}(E)$, $H_N^{\mu}(E)$, $H_{\alpha}^{\mu}(E,\lambda)$ and $H_{N\alpha}^{\mu}(E,\lambda)$ respectively.

Proof. We prove the result for the space $H^{\mu}(E)$, for the proof for other spaces being followed analogously.

Suppose $f \in H^{\mu}(E)$ and s is any scalar. Then the equality,

$$||\hat{d}^n(sf)(0)|| = ||s^n\hat{d}^n f(0)||^{1/n} = |s|\ ||\hat{d}^n f(0)||^{1/n},$$

yields the homogeneous character of Q_p. The triangle inequality, namely,

$$Q_p(f+g) \leq Q_p(f) + Q_p(g)$$

is a consequence of the inequality

(6.2.10) $(c+d)^n \leq c^n+d^n$, $0 < n < 1$, $c > 0$, $d > 0$

(cf. [152], p. 158).

Note: In view of Proposition 6.2.9, we can equip the spaces $H^{\mu}(E)$, $H_N^{\mu}(E)$, $H_{\alpha}^{\mu}(E,\lambda)$ and $H_{N\alpha}^{\mu}(E,\lambda)$ with locally convex topologies denoted throughout by T_h, T_h^N, $T_{h\alpha}$ and $T_{h\alpha}^N$, generated respectively by the families D_h, D_h^N, D_h^{λ} and $D_h^{N\lambda}$.

Concerning the spaces (6.2.1) and (6.2.2), we have

Proposition 6.2.11: Let (μ, T_{μ}) be a complete K-space such that $p_o(e^n) = 1$, for each $n \geq 0$ and some $p_o \in D_{\mu}$.

Then the space $(H_N^{\mu}(E),\ T_h^N)$ [resp. $(H^{\mu}(E), T_h)$] is quasi-complete.

Proof: For proving the quasi-completeness of the space $(H_N^{\mu}(E),\ T_h^N)$, consider a T_h^N-bounded Cauchy net $\{f_\beta : \beta \in I\}$ in $H_N^{\mu}(E)$. Fix $\varepsilon > 0$ and $p \in D_\mu$. Then there exists $\beta_o \equiv \beta_o(\varepsilon, p)$ such that

$$(+) \qquad p(\{(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N}{n!})^{1/n}\} < \varepsilon,\ \beta, \gamma \geq \beta_o.$$

Since p is monotone and

$$\left|(\frac{\|\hat{d}^n f_\beta(0)\|_N}{n!})^{1/n} - (\frac{\|\hat{d}^n f_\gamma(0)\|_N}{n!})^{1/n}\right| \leq (\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N}{n!})^{1/n}$$

by (6.2.10), it follows that

$$p(\{(\frac{\|\hat{d}^n f_\beta(0)\|_N}{n!})^{1/n} - (\frac{\|\hat{d}^n f_\gamma(0)\|_N}{n!})^{1/n}\}) < \varepsilon,\ \beta, \gamma \geq \beta_o$$

Hence the net $\{a^\beta : \beta \in I\}$, where

$$a^\beta = \{(\frac{\|\hat{d}^n f_\beta(0)\|_N}{n!})^{1/n}\},\ \beta \in I$$

is a Cauchy net in μ, which is complete. So there exists an element a in μ such that

$$a^\beta \to a \text{ in } T_\mu$$

$$(*) \qquad \Longrightarrow\ a_n^\beta \to a_n,\ \forall\, n \geq 0$$

Applying (+) for $p = p_o$ and using the monotone character of p_o, we get

$$p_o\left(\left(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N}{n!}\right)^{1/n} e^n\right) < \varepsilon, \ \beta, \gamma \geq \beta_o$$

$$\Longrightarrow \left(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N}{n!}\right)^{1/n} p_o(e^n) < \varepsilon, \ \beta, \gamma \geq \beta_o$$

But $p_o(e^n) = 1$, $n \geq 0$; therefore, the net $\{\hat{d}^n f_\beta(0) : \beta \in I\}$ is Cauchy in $(\mathcal{P}_N({}^nE), \|\cdot\|_N)$, for $n \geq 0$. Hence we can find a sequence $\{P_n\}$ of polynomials, $P_n \in \mathcal{P}_N({}^nE)$, $n \geq 0$ such that

$$(**) \qquad P_n = \lim_\beta \hat{d}^n f_\beta(0)$$

where limit taken is relative to the nuclear norm $\|\cdot\|_N$. Thus from (*) and (**), we get

$$a_n = \left(\frac{\|P_n\|_N}{n!}\right)^{1/n}, \ n \geq 0$$

Hence

$$\left\{\left(\frac{\|P_n\|_N}{n!}\right)^{1/n}\right\} \in \mu'$$

Let

$$Q_n(x) = \frac{P_n(x)}{n!}, \ n \geq 0$$

Clearly, $Q_n \in \mathcal{P}_N({}^nE)$, $n \geq 0$. We now show that the sequence $\left\{\left(\frac{\|Q_n\|}{n!}\right)^{1/n}\right\}$ is bounded, so that the series $\sum_{n\geq 0} Q_n(x)$ would converge in $\mathbb{C}$ (cf. Proposition 1.5.6) and would define a function f in $H_N^\mu(E)$ as follows:

$$(***) \qquad f(x) = \sum_{n \geq 0} Q_n(x), \ x \in E.$$

To prove this, observe that for the p_o of the hypothesis, there exists a constant $K \equiv K(p_o)$ such that

$$p_o(\{(\frac{\|\hat{d}^n f_\beta(0)\|_N}{n!})^{1/n}\}) \leq K, \ \forall \beta \in I,$$

since $\{f_\beta\}$ is bounded. Using the monotonicity of p_o and the fact that $p_o(e^n) = 1$, $n \geq 0$, we get

$$(\frac{\|\hat{d}^n f_\beta(0)\|_N}{n!})^{1/n} \leq K, \quad \forall n \geq 0, \ \beta \in I.$$

Hence passing to limit over β, we get

$$(\frac{\|P_n\|_N}{n!})^{1/n} \leq K, \ \forall n \geq 0.$$

Consequently,

$$(\frac{\|P_n\|}{n!})^{1/n} \leq (\frac{\|P_n\|_N}{n!})^{1/n} \leq K, \ \forall n \geq 0$$

$$\Longrightarrow \ (\frac{\|Q_n\|}{n!})^{1/n} = (\frac{\|P_n\|}{(n!)^2})^{1/n} \leq K, \ \forall n \geq 0.$$

Thus $f \in H^\mu(E)$. Since $\{(\frac{\|P_n\|_N}{n!})^{1/n}\} \in \mu$, $f \in H^\mu_N(E)$.

Finally, it remains to show that $f_\beta \to f$ in T^N_h. Define a net $\{b^\beta : \beta \in I\}$ in μ as follows:

$$b^\beta \equiv \{b^\beta_n\} = \{(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f(0)\|_N}{n!})^{1/n}\}, \beta \in I$$

Then from the monotonicity of each member of D_μ and the fact that

$$|b_n^\beta - b_n^\gamma| \leq \left(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N}{n!}\right)^{1/n}, \quad n \geq 0 ,$$

it follows from (+) that

$$p(b^\beta - b^\gamma) < \varepsilon, \ \beta, \gamma \geq \beta_0.$$

So $\{b^\beta : \beta \in I\}$ is a Cauchy net in (μ, T_μ). Hence there exists $b \in \mu$ such that

$$b^\beta \to b \ \text{ in } \ T_\mu$$

$$\Longrightarrow \quad b_n^\beta \to b_n, \ \forall\, n \geq 0$$

But

$$b_n^\beta = \left(\frac{\|\hat{d}^n f_\beta(0) - \hat{d}^n f(0)\|_N}{n!}\right)^{1/n}$$

$$\to 0, \text{ for each } n \geq 0.$$

Hence $b_n = 0$, $\forall\, n \geq 0$. Thus $b^\beta \to 0$ in (μ, T_μ), or equivalently, $f_\beta \to f$ in T_h^N. Hence the space $(H_N^\mu(E), T_h^N)$ is quasi-complete.

Proceeding exactly on similar lines, the quasi-completeness of the space $(H^\mu(E), T_h)$ also follows.

For our next result, we need to introduce

<u>Definition</u> 6.2.12: A sequence space λ is said to have G-<u>property</u> if λ contains an element a satisfying the condition

$$|a_n| > \frac{1}{|\alpha_n|(n!)^{1/n}}, \ \forall\, n \geq 0.$$

<u>Proposition</u> 6.2.13: Let (μ, T_μ) be as in Proposition 6.2.11, λ a sequence space with G-property and $\alpha \in \omega$ with $\alpha_n \neq 0$, $n \geq 0$. Then the spaces $(H^\mu_{N\alpha}(E,\lambda), T^N_{h\alpha})$ and $(H^\mu_\alpha(E,\lambda), T_{h\alpha})$ are quasi-complete.

<u>Proof</u>: We outline the proof for the quasi-completeness of $(H^\mu_{N\alpha}(E,\lambda), T^N_{h\alpha})$; the result for the space $(H^\mu_\alpha(E,\lambda), T_{h\alpha})$ being true on similar lines.

Let $\{f_\beta : \beta \in I\}$ be a $T^N_{h\alpha}$-bounded Cauchy net in $H^\mu_{N\alpha}(E,\lambda)$. Then for arbitrarily fixed $a \in \lambda$, $\varepsilon > 0$ and $p \in D_\mu$, there exists $\beta_0 \equiv \beta_0(\varepsilon, p, a)$ such that

$$(*) \qquad p(\{\, \|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N^{1/n} \alpha_n a_n\}) < \varepsilon,$$

for all $\beta, \gamma \geq \beta_0$. Write

$$\delta^\beta_a \equiv \{\delta^\beta_{a,n}\} = \{\, \|\hat{d}^n f_\beta(0)\|_N^{1/n} \alpha_n a_n\}, \ \beta \in I.$$

Due to the monotone character of p, we obtain

$$p(\{\|\hat{d}^n f_\beta(0)\|_N^{1/n} \alpha_n a_n\} - \{\|\hat{d}^n f_\gamma(0)\|_N^{1/n} \alpha_n a_n\}) < \varepsilon, \ \beta, \gamma \geq \beta_0$$

Hence the net $\{\delta^\beta_a : \beta \in I\}$, is a Cauchy net in (μ, T_μ) and so we get $\delta_a = \{\delta_{a,n}\} \in \mu$ such that

$$\delta_a = \lim_\beta \delta^\beta_a$$

On the other hand, choosing $p = p_o$ in (*), we get

$$p_o(\|\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)\|_N^{1/n} \alpha_n a_n e^n) < \varepsilon, \ \beta, \gamma \geq \beta_o.$$

But $p_o(e^n) = 1$ for each $n \geq 0$; therefore the net $\{\hat{d}^n f_\beta(0) : \beta \in I\}$ is a Cauchy net in $\mathcal{P}_N(^nE)$ for each fixed n. Consequently there is a sequence $\{P_n\}$ of nuclear polynomials, $P_n \in \mathcal{P}_N(^nE)$ such that

$$P_n = \lim_\beta \hat{d}^n f_\beta(0), \ n \geq 0$$

in the nuclear norm topology of $\mathcal{P}_N(^nE)$. Hence

$$\delta_{a,n} = \|P_n\|_N^{1/n} \alpha_n a_n, \ \forall n \geq 0$$

As $\delta_a \in \mu$ and $a \in \lambda$ is arbitrary, it follows that

$$\text{(++)} \qquad \{\|P_n\|_N^{1/n}\} \in \lambda_\alpha^\mu.$$

For $n \geq 0$, define

$$Q_n(x) = \frac{P_n(x)}{n!};$$

we now prove the boundedness of the sequence $\{(\frac{\|Q_n\|}{n!})^{1/n}\}$. Observe that for the a and p_o as given in the hypothesis, there exists a constant $K \equiv K(a, p_o)$ such that

$$\|\hat{d}^n f_\beta(0)\|_N^{1/n} p_o(\alpha_n a_n e^n) \leq K, \ \forall n \geq 0$$

Since $p_o(e^n) = 1$, $n \geq 0$ and

$$|a_n| > \frac{1}{|\alpha_n| (n!)^{1/n}} \quad , \ n \geq 0$$

by G-property of λ, we get

$$\left(\frac{||\hat{d}^n f_\beta(0)||_N}{n!}\right)^{1/n} \leq K, \ \forall \ n \geq 0$$

Hence

$$\left(\frac{||P_n||}{n!}\right)^{1/n} \leq \left(\frac{||P_n||_N}{n!}\right)^{1/n} \leq K, \ n \geq 0$$

$$\Longrightarrow \left(\frac{||Q_n||}{n!}\right)^{1/n} \leq K, \quad \forall \ n \geq 0 \ , \ \forall$$

Thus the function f defined by

$$f(x) = \sum_{n \geq 0} Q_n(x), \ x \in E$$

is a member of $H(E)$. Also it follows from (++) that

$$\{||P_n||_N^{1/n} \ \alpha_n a_n\} \in \mu \ , \ \forall \ a \in \lambda$$

and from the G-condition of λ, we have

$$\left(\frac{||P_n||_N}{n!}\right)^{1/n} \leq |\alpha_n a_n| \ ||P_n||_N^{1/n} \ , \ n \geq 0$$

for some fixed $a \in \lambda$. As μ is normal,

$$\left\{\left(\frac{||P_n||_N}{n!}\right)^{1/n}\right\} \in \mu$$

Consequently, $f \in H^{\mu}_{N\alpha}(E, \lambda)$

It remains to show that $f_\beta \to f$ in $T^N_{h\alpha}$. So it will be enough to show that, for each $a \in \lambda$ and $p \in D_\mu$, there exists $\beta_o \equiv \beta_o(\varepsilon, p, a)$ such that

$$(x) \qquad p(\{||\hat{d}^n f_\beta(0) - \hat{d}^n f(0)||_N^{1/n} \alpha_n a_n\}) < \varepsilon,\ \beta \geq \beta_o$$

where $\hat{d}^n f(0) = P_n$. To achieve this, let us write

$$\delta_a^\beta \equiv \{\delta_{a,n}^\beta\} = \{||\hat{d}^n f_\beta(0) - \hat{d}^n f(0)||_N^{1/n} \alpha_n a_n\} \in \mu, a \in \lambda, \beta \in I$$

Since

$$|\delta_{a,n}^\beta - \delta_{a,n}^\gamma| \leq ||\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)||_N^{1/n} \ |\alpha_n a_n|, n \geq 0$$

and p is monotone, it follows that

$$p(\delta_a^\beta - \delta_a^\gamma) \leq p(\{||\hat{d}^n f_\beta(0) - \hat{d}^n f_\gamma(0)||_N^{1/n} \alpha_n a_n \ )$$

$$< \varepsilon \ , \ \beta, \gamma \geq \beta_o$$

by (*). Therefore $\{\delta_a^\beta\}$ is a Cauchy net in μ, which is complete. Hence for some $\delta_a \equiv \{\delta_{a,n}\} \in \mu$ we get

$$\lim_\beta \ \delta_a^\beta = \delta_a$$

$$(.) \qquad \Longrightarrow \quad \lim_\beta \ \delta_{a,n}^\beta = \delta_{a,n}$$

Since $\hat{d}^n f_\beta(0) \to P_n = \hat{d}^n f(0)$, by (.) we have

$$\lim_\beta \delta_a^\beta = \delta_a = 0$$

Thus, the relation (x) holds and this establishes the result.

Remark: If μ, α and λ be as in (b) of the Note given after definition 5.1.1, the space $H^{\mu}_{N\alpha}(E,\lambda)$ and $H^{\mu}_{N}(E,\lambda)$ are the ones introduced and studied by Boland in [29], Section II (cf. Chapter 2 also) and therefore, our Proposition 6.2.13 includes his result, namely, the locally convex space $H_{Nb}(E,\mathcal{G})$ is a complete space, (cf. [29], Proposition 2.1, p. 49), as a particular case.

6.3. Bounded Sets

In this section we find several necessary and sufficient conditions for the characterization of bounded sets in the spaces $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$ and $(H^{\mu}_{\alpha}(E,\lambda), T_{h\alpha})$. As mentioned earlier, in the section two we consider sequences indexed over $\mathbb{N}_o$.

We begin our discussion for bounded subsets of $H^{\mu}_{N\alpha}(E,\lambda)$. For a subset B of $H^{\mu}_{N\alpha}(E,\lambda)$, let us introduce the notation

$$(6.3.1)\qquad \begin{aligned} b_o &= \sup\{|f(0)|:f \in B\}; \\ b^n_n &= \sup\{\|\hat{d}^n f(0)\|_N : f \in B\},\ n \geq 1. \end{aligned}$$

In the sequel, the symbol b whereever it is used, will stand for the sequence $\{b_n\}$ as introduced in (6.3.1).

To begin with, we have

Proposition 6.3.2: A subset B of $H^{\mu}_{N\alpha}(E,\lambda)$ is bounded if $b \in \lambda^{\mu}_{\alpha}$.

Proof: Let $p \in D_\mu$. Since for $a \in \lambda$ and $f \in B$

$$\|\hat{d}^n f(0)\|_N^{1/n} \; |\alpha_n a_n| \leq |b_n \alpha_n a_n| \; , \; \forall \; n \geq 0$$

and p is monotone, it follows that

$$Q^N_{a,p}(f) = p(\{\|\hat{d}^n f(0)\|_N^{1/n} \; \alpha_n a_n\}) \leq p(ab\alpha), \; \forall \; f \in B$$

Hence B is bounded.

The situation for the validity of the converse of Proposition 6.3.2 is not so pleasant in general; however, restriction on μ or on both the spaces μ and λ, leads to the following three different variations of the converse

Proposition 6.3.3: Let B be a bounded subset of $H^\mu_{N\alpha}(E,\lambda)$. If μ contains e and $p_o(e^n) = 1$, $n \geq 0$, for some $p_o \in D_\mu$, then $b \in \lambda^\mu_\alpha$.

Proof: Since B is bounded, for p_o as in hypothesis and $a \in \lambda$, there exists a constant $K \equiv K(a,p_o)$ such that

$$Q^N_{a,p_o}(f) = p_o(\{\|\hat{d}^n f(0)\|_N^{1/n} \; \alpha_n a_n\}) \leq K, \; \forall \; f \in B$$

$$\Longrightarrow p_o(\{b_n \alpha_n a_n\}) < K$$

$$\Longrightarrow |\alpha_n a_n b_n| \leq K, \; \forall \; n \geq 0$$

as p_o is monotone and $p_o(e^n) = 1$, $n \geq 0$. But μ is normal and contains constant sequences; therefore $\alpha ab \in \mu$ for each a in λ. Consequently, $b \in \lambda^\mu_\alpha$. This completes the proof.

For our next result, we make use of

<u>Lemma</u> 6.3.4. If λ is a normal sequence space and $(\lambda^x, \eta(\lambda^x, \lambda))$ is nuclear, then

$$\lambda^x = \{c \in \omega : \sup_n |c_n a_n| < \infty, \ \forall\, a \in \lambda\}$$

$$= \{c \in \omega : |c_n a_n| \to 0 \quad \text{as} \quad n \to \infty, \ \forall\, a \in \lambda\}.$$

<u>Proof:</u> By definition of λ^x, we have

$$\lambda^x = \{c \in \omega : \sum_{n \geq 1} |c_n a_n| < \infty, \ \forall\, a \in \lambda\}$$

Let us write

$$\lambda^1 = \{c \in \omega : \sup_n |c_n a_n| < \infty \ , \forall\, a \in \lambda\}$$

and

$$\lambda^2 = \{c \in \omega : |c_n a_n| \to 0 \text{ as } n \to \infty, \ \forall\, a \in \lambda\}$$

Clearly,

$$\lambda^x \subset \lambda^2 \subset \lambda^1$$

In order to prove the equality, we have to show that

$$\lambda^1 \subset \lambda^x.$$

Therefore, consider $c \in \lambda^1$. Take any a in λ, so by Proposition 1.3.8, there exists an $e \in \lambda$ such that $\{a_n/e_n\} \in \ell^1$. But

$$\sum_{n \geq 1} |c_n a_n| = \sum_{n \geq 1} |c_n a_n/e_n, \ e_n|$$

$$\leq \sup_{n \geq 1} |c_n e_n| \cdot \sum_{n \geq 1} \left|\frac{a_n}{e_n}\right|$$

$$< \infty$$

Proposition 6.3.5: Let B be a bounded subset of $H^{\mu}_{N\alpha}(E,\lambda)$. If $(\mu^{x},\eta(\mu^{x},\mu))$ is a perfect nuclear sequence space, then $b \in \lambda^{\mu}_{\alpha}$.

Proof: If $b \notin \lambda^{\mu}_{\alpha}$, then there exists a in λ, with the property that $\alpha ab \notin \mu$. Using the perfectness of μ and Lemma 6.3.4, we can find c in μ^{x} such that

$$\sup_{n} |a_n b_n c_n \alpha_n| = \infty$$

Therefore, for each $i \geq 0$, there exists a subsequence $\{n_i\}$ of $\mathbb{N}$ such that

$$|a_{n_i} b_{n_i} c_{n_i} \alpha_{n_i}| > 2^{i} \quad , \; i \geq 0$$

or,
$$\sup_{f \in B} ||\hat{d}^{n_i} f(0)||_N^{1/n_i} \; |a_{n_i} c_{n_i} \alpha_{n_i}| > 2^{i}, \; i \geq 0$$

Consequently, there exists a sequence $\{f_i\}$ in B such that

$$||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} \; |\alpha_{n_i} a_{n_i} c_{n_i}| > 2^{i}, \; i \geq 0$$

$$\Longrightarrow \; Q^{N}_{c,a}(f_i) = \sum_{n \geq 0} ||\hat{d}^{n} f_i(0)||_N^{1/n} \; |a_n c_n \alpha_n|.$$

$$\geq \; ||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} \; |\alpha_{n_i} a_{n_i} c_{n_i}|$$

$$> \; 2^{i}, \; i \geq 0$$

This contradicts the boundedness of B in $H^{\mu}_{N\alpha}(E,\lambda)$. Hence $b \in \lambda^{\mu}_{\alpha}$.

<u>Proposition</u> 6.3.6: Let λ be a normal sequence space such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz and suppose that $\mu = c_o$. Then $b \in \lambda^\mu_\alpha$, if B is a bounded set in $(H^\mu_{N\alpha}(E,\lambda),\ T^N_{h\alpha})$.

<u>Proof</u>: Assume that $b \notin \lambda^\mu_\alpha$. Then we can find $a \in \lambda$ such that $\alpha ab \notin c_o$. Hence there is an $\varepsilon > 0$ and an increasing sequences $\{n_i\}$ for which

$$|\alpha_{n_i} a_{n_i} b_{n_i}| > \varepsilon,\ \forall\ i \geq 0$$

Consequently, we get a sequence $\{f_i\} \subset B$ satisfying

$$||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} |\alpha_{n_i} a_{n_i}| > \varepsilon,\ \forall\ i \geq 0.$$

By Proposition 1.3.9, there exists a sequence $e \in \lambda$ such that

$$\{a_n/e_n\} \in c_o$$

Write $\beta_n = a_n/e_n$, $n \geq 0$. Then for each $i \geq 0$,

$$\begin{aligned} Q^N_{||\cdot|| e}(\beta_{n_i} f_i) &= \sup_{j\geq 0} ||\hat{d}^j(\beta_{n_i} f_i)(0)||_N^{1/j} |\alpha_j e_j| \\ &\geq ||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} |\beta_{n_i} \alpha_{n_i} e_{n_i}| \\ &= ||\hat{d}^{n_i} f_i(0)||_N^{1/n_i} |\alpha_{n_i} a_{n_i}| \\ &> \varepsilon \end{aligned}$$

Hence $\beta_{n_i} f_i \not\to 0$ in $T^N_{n\alpha}$. This contradicts the boundedness of B and so the result holds good.

To characterize bounded subsets of $H^{\mu}_{\alpha}(E,\lambda)$, for a subset D of $(H^{\mu}_{\alpha}(E,\lambda), T_{h\alpha})$, let us write

$$d_o = \sup \{|f(0)| : f \in D\}$$

(6.3.6)*

$$d^n_n = \sup \{\|\hat{d}^n f(0)\| : f \in D\}$$

Let us fix the symbol d to denote the sequence $\{d_n : n \geq 0\}$ as defined above. Then we have

<u>Proposition</u> 6.3.7. A subset D of $H^{\mu}_{\alpha}(E,\lambda)$ is bounded if $d \in \lambda^{\mu}_{\alpha}$.

<u>Proof</u>: Following the lines of proof of Proposition 6.3.2, we get this result.

<u>Proposition</u> 6.3.8: Let D be a bounded subset of $H^{\mu}_{\alpha}(E,\lambda)$. If μ contains e and $p_o(e^n) = 1$, $n \geq 0$, for some $p_o \in D_{\mu}$, then $d \in \lambda^{\mu}_{\alpha}$.

<u>Proof</u>: The proof is analogous to that of Proposition 6.3.3.

<u>Proposition</u> 6.3.9: Let $(\mu^x, \eta(\mu^x, \mu))$ be a perfect nuclear sequence space. Then for a bounded subset D of $H^{\mu}_{\alpha}(E,\lambda)$, $d \in \lambda^{\mu}_{\alpha}$.

<u>Proof</u>: Its proof is similar to that of Proposition 6.3.5.

<u>Proposition</u> 6.3.10: Let λ be a normal sequence space such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz and suppose that $\mu = c_o$. Then $d \in \lambda^{\mu}_{\alpha}$ if D is a bounded set in $(H^{\mu}_{\alpha}(E,\lambda), T_{h\alpha})$.

Proof: Proceeding on the lines of Proposition 6.3.6, we get the result.

6.4 The Induced Topologies:

Since the spaces $H^{\mu}_{\alpha}(E,\lambda)$ and $H^{\mu}_{N\alpha}(E,\lambda)$ are respectively contained in $H^{\mu}(E)$ and $H^{\mu}_{\alpha}(E)$, it is natural to inquire the relationship between the original and induced topologies.

Essentially, we investigate conditions under which the two topologies coincide on bounded subsets of $H^{\mu}_{\alpha}(E,\lambda)$ and $H^{\mu}_{N\alpha}(E,\lambda)$. In this direction, we have

Proposition 6.4.1: Let λ possess G-property. Then the induced topology $T_H/H^{\mu}_{\alpha}(E,\lambda)$ on $H^{\mu}_{\alpha}(E,\lambda)$ [resp. $T_{h\alpha}/H^{\mu}_{N\alpha}(E,\lambda)$ on $H^{\mu}_{N\alpha}(E,\lambda)$] is weaker than the topology $T_{h\alpha}$[resp. $T^{N}_{h\alpha}$].

Proof: Since each p in D_{μ} is monotone and

$$\left(\frac{\|\hat{d}^n f(0)\|_N}{n!}\right)^{1/n} \leq \|\hat{d}^n f(0)\|_N^{1/n} |\alpha_n a_n|$$

for each $f \in H^{\mu}_{N\alpha}(E,\lambda)$, $n \geq 0$ and some $a \in \lambda$, the result follows from the definition of seminorms $Q_p, Q_{p,a}$ (resp. Q^N_p, $Q^N_{p,a}$) [cf. 6.2.5, 6.2.7 6.2.6, 6.2.8]

On the other hand, on bounded subsets we have

Proposition 6.4.2: Let λ be a normal sequence space such that $(\lambda^{x},\eta(\lambda^{x},\lambda))$ is Schwartz and $\mu=c_0$. If

B[resp. D] is a bounded subset of $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$ [resp. $(H^{\mu}_{\alpha}(E,\lambda),\ T_{h\alpha})$], then the topologies induced on B [resp. D] by $T^{N}_{h\alpha}$ and T^{N}_{h} [resp. $T_{h\alpha}$ and T_{h}] coincide, provided λ satisfies G-property.

<u>Proof</u>: We prove the result for $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$; the result for bracketed space follows on similar lines.

In view of Proposition 6.4.1, we need prove

$$(+) \qquad T^{N}_{h\alpha}|B \subset T^{N}_{h}|B$$

For proving (+), let us take a net $\{f_{\delta}:\delta \in I\} \subset B$ such that $f_{\delta} \to f$ in T^{N}_{h}, for some $f \in B$. Then for given $\varepsilon > 0$, there exists $\delta_{o} \equiv \delta_{o}(\varepsilon)$ in I such that

$$\sup_{n} \{ (\frac{||\hat{d}^{n}f_{\delta}(0)-\hat{d}^{n}f(0)||_{N}}{n!})^{1/n} \} < \varepsilon,\ \delta \geq \delta_{o}$$

$$(*) \qquad \Longrightarrow \quad \hat{d}^{n}f_{\delta}(0) \to \hat{d}^{n}f(0) \quad \text{in } \mathcal{P}_{N}(^{n}E),\ \forall\ n \geq 0.$$

Also, $b \in \lambda^{c_o}_{\alpha}$ by Proposition 6.3.6, since B is bounded. Therefore for $a \in \lambda$, $ab\alpha \in c_{o}$ and consequently there exists an integer $n_{o} \equiv n_{o}(\varepsilon, a)$ such that

$$|a_{n}b_{n}\alpha_{n}| < \varepsilon/2 \quad ,\ n \geq n_{o}.$$

Hence

$$\begin{aligned} Q^{N}_{||\cdot||,a}(f_{\delta}-f) &= \sup_{n} ||\hat{d}^{n}f_{\delta}(0)-\hat{d}^{n}f(0)||_{N}^{1/n} |\alpha_{n}a_{n}| \\ &\leq \quad 2 \sup_{n} |a_{n}b_{n}\alpha_{n}| \\ &< \quad \varepsilon\ , \quad n \geq n_{o} \end{aligned}$$

Using (*), for sufficiently large δ, say $\delta \geq \delta_o^*$, we get

$$\sup_{n \leq n_o - 1} \| \hat{d}^n f_\delta(0) - \hat{d}^n f(0) \|_N^{1/n} |\alpha_n a_n| < \varepsilon.$$

Hence for $\delta \geq \delta_o^*$,

$$Q_{\|\cdot\|, a}(f_\delta - f) \leq \sup_{n \leq n_o - 1} \{ \| \hat{d}^n f_\delta(0) - \hat{d}^n f(0) \|_N^{1/n} |\alpha_n a_n|, \varepsilon \}$$
$$< \varepsilon$$

So $f_\delta \to f$ in $T_{h\alpha}^N$. This proves (+).

Proposition 6.4.3: Let B[resp. D] be a bounded subset of $(H_{N\alpha}^\mu(E, \lambda), T_{h\alpha}^N)$ [resp. $(H_\alpha^\mu(E, \lambda), T_{h\alpha})$]. Then the topologies induced on B[resp. on D] by $T_{h\alpha}^N$ and T_h^N [resp. $T_{h\alpha}$ and T_h] coincide, if λ has G-property and $(\mu^x, \eta(\mu^x, \mu))$ is a perfect nuclear sequence space.

Proof: In view of Proposition 6.4.1, it is enough to show that $f_\delta \to f$ in $T_{h\alpha}^N$ whenever $f_\delta \to f$ in T_h^N, where the net $\{f_\delta : \delta \in I\}$ and the element f are taken from B.

Let us therefore consider $\{f_\delta : \delta \in I\}$ and f as above. Observe that $b \in \lambda_\alpha^\mu$ by Proposition 6.3.5. Hence, for $\varepsilon > 0$, $a \in \lambda$ and $c \in \mu^x$, there exists an integer $N_o \equiv N_o(\varepsilon)$ such that

(+) $$\sum_{n \geq N_o} |b_n \alpha_n a_n c_n| < \varepsilon/4$$

Also, there exists an index $\delta_o \equiv \delta_o(\varepsilon, c)$ such that for $\delta \geq \delta_o$,

$$\sum_{n\geq 0} ||\hat{d}^n f_\delta(0)-\hat{d}^n f(0)||_N^{1/n} \; |c_n| < \varepsilon$$

$$\Longrightarrow \; ||\hat{d}^n f_\delta(0)-\hat{d}^n f(0)||_N^{1/n} |c_n| < \varepsilon, \; \delta \geq \delta_0, \; \forall \, n \geq 0$$

$$\Longrightarrow \; \hat{d}^n f_\delta(0) \to \hat{d}^n f(0) \; \text{ in } \mathcal{P}_N(^nE), \; \forall \, n \geq 0.$$

Thus, there exists an index δ_0^* in I such that

$$(*) \qquad \sum_{n=0}^{N_0-1} ||\hat{d}^n f_\delta(0)-\hat{d}^n f(0)||_N^{1/n} \; |\alpha_n a_n c_n| < \varepsilon/2$$

for $\delta \geq \delta_0^*$. Hence by (+) and (*) we get

$$\begin{aligned} Q^N_{a,c}(f_\delta - f) &= \sum_{n\geq 0} ||\hat{d}^n f_\delta(0)-\hat{d}^n f(0)||_N^{1/n} |\alpha_n a_n c_n| \\ &\leq \sum_{n=0}^{N_0-1} ||\hat{d}^n f_\delta(0)-\hat{d}^n f(0)||_N^{1/n} |\alpha_n a_n c_n| \\ &\qquad + 2\sum_n |b_n \alpha_n a_n c_n| \\ &< \varepsilon \end{aligned}$$

for $\delta \geq \delta_0^*$. Therefore, $f_\delta \to f$ in $T^N_{h\alpha}$.

The proof for the bracketed space follows analogously.

6.5 Relatively Compact Sets:

In this section we characterize the relatively compact sets in the spaces $H^\mu_{N\alpha}(E,\lambda)$ and $H^\mu_\alpha(E,\lambda)$ and make frequent use of the results of the preceding section. Indeed, we reserve the symbols b and d to denote the sequences defined by (6.3.1) and (6.3.6)* for the sets B and D in $H^\mu_{N\alpha}(E,\lambda)$ and $H^\mu_\alpha(E,\lambda)$ respectively. Let us begin with

<u>Proposition</u> 6.5.1: Let $(\mu^{x}, \eta(\mu^{x}, \mu))$ be a perfect nuclear sequence space and λ possess the G-property. Then a set B [resp. D] in $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$ [resp. in $(H^{\mu}_{\alpha}(E,\lambda), T_{h\alpha})$] is relatively compact if and only if B [resp. D] is bounded and the set $\{\hat{d}^{n}f(0): f \in B\}$ [resp. $\{\hat{d}^{n}f(0): f \in D\}$] is relatively compact in $\mathcal{P}_{N}({}^{n}E)$ [resp. in $\mathcal{P}({}^{n}E)$] for each $n \geq 0$.

<u>Proof</u>: We prove the result for the space $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$, the result for the space $(H^{\mu}_{\alpha}(E,\lambda), T_{h\alpha})$ follows on similar lines.

Assume that B is not relatively compact. As the space $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$ is quasicomplete by Proposition 6.2.1, B is not precompact. Hence there exist $a \in \lambda$, $c \in \mu^{x}$, $\varepsilon > 0$ and a sequence $\{f_i\}$ in B such that

$$Q^{N}_{a,c}(f_i - f_j) > \varepsilon \quad , \ \forall\ i,j \geq 0,$$

$$(*) \qquad \text{or,} \quad \sum_{n \geq 0} \|\hat{d}^{n}f_i(0) - \hat{d}^{n}f_j(0)\|_{N}^{1/n} |\alpha_n a_n c_n| < \varepsilon, \ \forall\ i,j \geq 0$$

Also, $b \equiv \{b_n\} \in \lambda^{\mu}_{\alpha}$ by Proposition 6.3.5; therefore for the above $\varepsilon > 0$, there exists an integer $n_o \equiv n_o(\varepsilon)$ such that

$$\sum_{n \geq n_o+1} |\alpha_n a_n b_n c_n| < \varepsilon/4.$$

Consequently,

$$\sum_{n \geq n_o+1} ||\hat{d}^n f_i(0)-\hat{d}^n f_j(0)||_N^{1/n} |\alpha_n a_n c_n|$$

$$\leq 2 \sum_{n \geq n_o+1} |\alpha_n a_n b_n c_n| < \varepsilon/2, \quad \forall\, i,j \geq 0.$$

Hence, we have from (*)

$$\sum_{n=0}^{n_o} ||\hat{d}^n f_i(0)-\hat{d}^n f_j(0)||_N^{1/n} |\alpha_n a_n c_n| > \varepsilon/2, \quad \forall\, i,j \geq 0.$$

Thus for each pair $(i,j) \subset \mathbb{N}_o \times \mathbb{N}_o$, there exists an integer n_{ij} lying between 0 and n_o such that

$$(+) \qquad ||\hat{d}^{n_{ij}} f_i(0)-\hat{d}^{n_{ij}} f_j(0)||_N^{1/n} |\alpha_n a_n c_n| > \frac{\varepsilon}{2(n_o+1)}$$

Consequently, the inequality (+) is satisfied for infinitely many i,j's corresponding to the same $n \equiv n_{ij}$ lying between 0 and n_o. This contradicts the relative compactness of the set $\{\hat{d}^n f(0): f \in B\}$ for each $n \geq 0$.

Conversely, if B is relatively compact, then B is bounded. For $n \geq 0$, define linear maps $\Psi_n : H^{\mu}_{N\alpha}(E,\lambda) \to \mathcal{P}_N($ as follows:

$$\Psi_n(f) = \hat{d}^n f(0), \quad n \geq 0$$

In order to show the continuity of Ψ_n, for each $n \geq 0$, let us consider a net $\{f_\beta : \beta \in I\}$ in $H^{\mu}_{N\alpha}(E,\lambda)$ such that $f_\beta \to f$ in $T^N_{h\alpha}$, for some $f \in H^{\mu}_{N\alpha}(E,\lambda)$. Then for a given $\varepsilon > 0$, $a \in \lambda$ and $c \in \mu^x$, there exists $\beta_o \equiv \beta_o(\varepsilon,a,c)$ such that

$$Q^{N}_{a,c}(f_\beta - f) = \sum_{n \geq 0} \|\hat{d}^n f_\beta(0) - \hat{d}^n f(0)\|_N^{1/n} |\alpha_n a_n c_n|$$

$$< \varepsilon, \forall \beta \geq \beta_o$$

$$\Longrightarrow \|\hat{d}^n f_\beta(0) - \hat{d}^n f(0)\|_N^{1/n} |\alpha_n a_n c_n| < \varepsilon, \forall n \geq 0, \beta \geq \beta_o$$

$$\Longrightarrow \lim_\beta \hat{d}^n f_\beta(0) = \hat{d}^n f(0), \quad \forall n \geq 0$$

$$\text{or} \quad \lim_\beta \Psi_n(f_\beta) = \Psi_n(f), \forall n \geq 0.$$

Thus Ψ_n is continuous for each $n \geq 0$. Hence the sets $\{\hat{d}^n f(0) : f \in B\}$, $n \geq 0$, being the continuous images of relatively compact set B, under Ψ_n, are relatively compact. This completes the proof.

Finally, we prove

Proposition 6.5.2: Let λ be a normal sequence space with G-property such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is Schwartz and $\mu = c_o$. Then $B \subset H^{\mu}_{N\alpha}(E,\lambda)$ [resp. $D \subset H^{\mu}_{\alpha}(E,\lambda)$] is $T^N_{h\alpha}$-[resp. $T_{h\alpha}$-] relatively compact if and only if B is $T^N_{h\alpha}$-[resp. $T_{h\alpha}$-] bounded and the set $\{\hat{d}^n f(0) : f \in B\}$ [resp. $\{\hat{d}^n f(0) : f \in D\}$] is relatively compact in $\mathcal{P}_N({}^nE)$ [resp. $\mathcal{P}({}^nE)$] for each $n \geq 0$.

Proof: The proof of this result is not different from that of the preceding one, however, for the sake of completeness we outline the same.

If B is not relatively compact, then it is not precompact as $(H^{\mu}_{N\alpha}(E,\lambda), T^{N}_{h\alpha})$ is quasi-complete by Proposition 6.2.13. So there exist $\varepsilon > 0$, $a \in \lambda$ and a sequence $\{f_i\}$ in B such that

$$(+) \qquad \sup_n \|\hat{d}^n f_i(0) - \hat{d}^n f_j(0)\|_N^{1/n} |\alpha_n a_n| > \varepsilon, \ \forall\ i,j \geq 0$$

But, by Proposition 6.3.5, we have $b \equiv \{b_n\} \in \lambda_\alpha^{c_0}$ and so $\{b_n a_n \alpha_n\} \in c_0$. Hence, there exists an integer $n_0 \equiv n_0(\varepsilon)$ such that

$$|b_n a_n \alpha_n| < \varepsilon, \ n \geq n_0.$$

Therefore, from (+) we infer that

$$\sup_{0 \leq n \leq n_0 - 1} \|\hat{d}^n f_i(0) - \hat{d}^n f_j(0)\|_N^{1/n} |\alpha_n a_n| > \varepsilon, \ \forall\ i,j \geq 0$$

Consequently, for each pair $(i,j) \subset \mathbb{N}_0 \times \mathbb{N}_0$, there exists integer n_{ij}, $0 \leq n_{ij} \leq n_0 - 1$ such that

$$(*) \qquad \|\hat{d}^{n_{ij}} f_i(0) - \hat{d}^{n_{ij}} f_j(0)\|_N^{1/n} |\alpha_n a_n| > \varepsilon$$

Thus, there exists $n = n_{ij}$, $0 \leq n_{ij} \leq n_0 - 1$, for which (*) is valid for infinitely many i,j's, which contradicts the fact that, the set $\{\hat{d}^n f(0) : f \in B\}$ is relatively compact for each $n \geq 0$.

Conversely, for $n \geq 0$, define linear maps $\Psi_n : H^{\mu}_{N\alpha}(E,\lambda) \to \mathcal{P}_N({}^nE)$ as in the proof of preceding

proposition. For continuty of Ψ_n, observe that if $f_\beta \to f$ in $H^{\mu}_{N\alpha}(E,\lambda)$ relative to $T^N_{h\alpha}$, then for $\varepsilon > 0$, $a \in \lambda$, there exists $\beta_o \equiv \beta_o(\varepsilon, a)$ such that

$$\sup_n \left| |\hat{d}^n f_\beta(0) - \hat{d}^n f(0)| \right|_N^{1/n} |\alpha_n a_n| < \varepsilon, \ \beta \geq \beta_o$$

$$\Longrightarrow \quad \hat{d}^n f_\beta(0) \to \hat{d}^n f(0), \ \forall\, n \geq 0$$

$$\text{or} \quad \Psi_n(f_\beta) \to \Psi_n(f), \ \forall\, n \geq 0.$$

Hence the sets $\{\hat{d}^n f(0) : f \in B\} = \Psi_n(B)$, $n \geq 0$, are relatively compact for each $n \geq 0$.

The proof for the other space $H^{\mu}_{\alpha}(E,\lambda)$ follows analogously.

Chapter 7

Holomorphic Functions on Nuclear Sequence Space

7.1 Introduction

In this chapter we take up the study of a class of holomorphic (indeed hypoanalytic) mappings defined on an open subset of the dual of a nuclear sequence space. Besides obtaining several results on nuclear sequence spaces, we finally get the Schauder basis representation of elements of this class with respect to the compact open topology.

7.2 Some Results on Nuclear Sequence Spaces

In this section we prove some results involving open subsets of the dual of nuclear sequence spaces. We will use these results in the proof of the main result, to be proved in the next section. At the outset, let us prove

Proposition 7.2.1: Let λ be a normal sequence space such that $(\lambda^x, \eta(\lambda^x, \lambda))$ is nuclear. For a neighbourhood U of zero in $(\lambda^x, \eta(\lambda^x, \lambda))$, there exists an absolutely convex neighbourhood v of zero and a sequence $\alpha \equiv \{\alpha_n\}$ with $\alpha_n > 1$, for each $n \geq 1$ and $\{\alpha_n^{-1}\} \in \ell^1$ such that

$$\alpha v = \{\{\alpha_n b_n\}: \{b_n\} \in v,\ \{\alpha_n b_n\} \in \lambda^x\}$$
$$\subset U$$

<u>Proof</u>: In view of Lemma 6.3.4, we may assume

$$U = \{b \in \lambda^x: \sup_n |b_n a_n| < \varepsilon\},$$

for some $a \equiv \{a_n\} \in \lambda$ and $\varepsilon > 0$. Using Proposition 1.3.8 we can find $c \equiv \{c_n\}$ in λ with $0 \leq a_n \leq c_n$ such that $\{a_n/c_n\} \in \ell^1$. Define

$$v = \{b \in \lambda^x: \sup_n |b_n c_n| < \varepsilon\}$$

and

$$\alpha_n = \begin{cases} 2^n & \text{if } a_n = 0 \\ c_n/a_n & \text{if } a_n \neq 0 \end{cases}$$

Clearly, $\alpha_n > 1$ for each $n \geq 1$ and $\{1/\alpha_n\} \in \ell^1$. Also for $\{b_n\} \in v$, we have

$$\sup_n |\alpha_n b_n a_n| \leq \sup_n |b_n c_n| < \varepsilon.$$

Hence $\alpha v \subset U$. This completes the proof.

<u>Proposition</u> 7.2.2: Let $(\lambda, \eta(\lambda, \lambda^x))$ be a barrelled nuclear sequence space and U a normal open set in $(\lambda^x, \beta(\lambda^x, \lambda))$. If K is a compact subset of U, then there exists a sequence $\delta \equiv \{\delta_n\}$ such that $\delta_n > 1$ for each $n \geq 1$, $\{1/\delta_n\} \in \ell^1$ and

$$\delta K = \{\{\delta_n a_n\}: a = \{a_n\} \in K\}$$

is a relatively compact subset of U.

Proof: We may assume without loss of generality that K is normal. Since $(\lambda, \eta(\lambda, \lambda^x))$ is barrelled every bounded subset of $(\lambda^x, \beta(\lambda^x, \lambda))$ is equicontinuous (cf. [119], Proposition 3.6.6, p. 217) and so there exists a positive sequence $b \equiv \{b_n\}$ in λ^x such that $K \subset w^o$, where

$$w = \{a \in \lambda: \sum_{n \geq 1} |a_n b_n| \leq 1\}$$

But

$$(*) \quad w^o = \{a \in \lambda: \sum_{n>1} |a_n b_n| \leq 1\}^o = \{a \in \lambda^x: |a_n| \leq |b_n|, \ \forall\, n \geq 1\}.$$

indeed, if $c \in w^o$, then $|\sum_{n \geq 1} c_n a_n| \leq 1$, $\forall\, a \in w$. Then for $a \in \lambda$,

$$|\sum_{n \geq 1} c_n a_n| \leq \sum_{n \geq 1} |a_n b_n|$$

In particular, taking $a = e^n$, $|c_n| \leq |b_n|$, $\forall\, n \geq 1$. Let d corresponds to b in λ^x such that $\{b_n/d_n\} \in \ell^1$ (cf. Proposition 1.3.8). Define a neighbourhood v of origin as follows

$$v = \{a \in \lambda: \sum_{n \geq 1} |a_n d_n| \leq 1\}$$

Now for each $x \in K \subset U$, there exists $\beta(\lambda^x, \lambda)$-neighbourhood v_x of origin such that

$$x + v_x \subset U$$

Choose $\beta(\lambda^x, \lambda)$-neighbourhoods w_x of origin such that

$$w_x + w_x \subset v_x$$

Since the family $\{x+w_x : x \in K\}$ is an open cover of the compact set K in the topology $\beta(\lambda^x, \lambda)$, there exists $x_1, \ldots, x_n \in K$ such that

$$K \subset \bigcup_{i=1}^{n} (x_i + w_{x_i})$$

Let

$$N = \bigcap_{i=1}^{n} w_{x_i}$$

As N is a $\beta(\lambda^x, \lambda)$-neighbourhood of origin, there exists a $\sigma(\lambda, \lambda^x)$-bounded and so $\eta(\lambda, \lambda^x)$ bounded set A in λ with $A^o \subset N$. Also, there exists $\varepsilon > 0$ such that

$$\begin{aligned} & \varepsilon A \subset v \\ \Longrightarrow \quad & \varepsilon v^o \subset A^o \subset N \\ \Longrightarrow \quad & K+\varepsilon v^o \subset K+N \end{aligned}$$

But

$$\begin{aligned} K+N & \subset \bigcup_{i=1}^{n} (x_i + w_{x_i}) + w_{x_i} \\ & \subset \bigcup_{i=1}^{n} (x_i + w_{x_i} + w_{x_i}) \\ & \subset \bigcup_{i=1}^{n} (x_i + v_{x_i}) \\ & \subset U \end{aligned}$$

Hence

$$K+\varepsilon v^o \subset U.$$

Further, observe that $K+\varepsilon v^o$ is a relatively compact subset of U (cf. [119], p. 212). Now define $\delta \equiv \{\delta_n\}$ as follows:

$$\delta_n = \begin{cases} 1+\varepsilon \dfrac{d_n}{b_n} , & \text{if } b_n \neq 0 ; \\ 2^n, & \text{otherwise} \end{cases}$$

Clearly, $\delta_n > 1$ for each $n \geq 1$ and $\{1/\delta_n\} \in \ell^1$, for we have,

$$\sum_{n\geq 1} \frac{1}{\delta_n} < \sum_{n\geq 1} \frac{1}{1+\varepsilon\, d_n/b_n} + \sum_{n\geq 1} \frac{1}{2^n}$$

$$< \frac{1}{\varepsilon} \sum_{n\geq 1} \frac{b_n}{d_n} + 1 < \infty$$

Since

$$V^o = \{a \in \lambda^x \colon |a_n| \leq |d_n| , \forall\, n \geq 1\},$$

(cf. proof of (*) as given above), it follows from the normality of K that

$$\delta K \subset K+\varepsilon \{a \in \lambda^x \colon |a_n| \leq |d_n|, n \geq 1\} \subset U,$$

indeed, for $a \equiv \{a_n\} \in K$,

$$|\delta_n a_n| \leq |a_n| + \varepsilon |d_n| , \forall\, n \geq 1.$$

This completes the proof.

7.3 The Main Result

In order to state the main result of this chapter, let us first recall from Section 5, Chapter 1, the definition of G-holomorphic mapping defined on an open subset U of an l.c. TVS (X, T) and then introduce the following

Definition 7.3.1: A mapping $f: U \to \mathbb{C}$, where U is an open subset of an l.c. TVS(X,T), is said to be hypoanalytic if f is G-holomorphic and continuous on compact subsets of U. We denote the collection of all hypoanalytic functions from $U \to \mathbb{C}$, by the symbol $H_{HY}(U)$. Clearly $H_{HY}(U)$ is a vector space with usual pointwise addition and scalar multiplication. Moreover, it is clear that

$$H(U) \subset H_{HY}(U)$$

where $H(U)$ is the space of all continuous G-holomorphic functions on U (cf. Definition 1.5.11).

Recalling the notation τ_o from Section 5 which is nothing but the compact open topology, the monomials $\{f^m, m \in \mathbb{N}^{(\mathbb{N})}\}$ from Section 4 and the definition of fully λ-bases from Section 2 of Chapter 1, we now state the main result contained in

Theorem 7.3.2: Let $(\lambda, \eta(\lambda, \lambda^x))$ be a barrelled nuclear sequence space such that $(\lambda^x, \beta(\lambda^x, \lambda))$ is an AK-space. Then the class $H_{HY}(U)$ of hypoanalytic functions defined on a normal open subset U of $(\lambda^x, \beta(\lambda^x, \lambda))$ equipped with the topology τ_o of uniform convergence on compact subsets of U, is a complete nuclear space and the set $\{f^m: m \in \mathbb{N}^{(\mathbb{N})}\}$ of monomials forms a fully ℓ^1-base for $(H_{HY}(U), \tau_o)$.

The proof of this theorem requires a preparatory result which we prove in much more generality in the form of

<u>Proposition</u> 7.3.3: Let $\{x_n, f_n\}$ be a fully λ-base for an l.c. TVS (X, T), where λ satisfies the (K)-property (i.e. there exists c in λ^x with $K_c \equiv \inf c_n > 0$). Then there exists a Köthe set P such that $(X,T) \simeq (\mu, T_P/\mu)$, where $\mu = \{\{f_n(x)\}: x \in X\}$ is a dense subspace of $(\Lambda(P), T_P)$; in particular, if (X,T) is sequentially complete, then $(X,T) \simeq (\Lambda(P), T_P)$.

<u>Proof</u>: Let

$$P = \{\{p(x_n)\beta_n\}: p \in D_T, \ \beta \in \lambda^x, \ \beta > 0\}$$

and $\Lambda(P)$ the corresponding Köthe space equipped with the topology T_P. Since for each $p \in D_T$, the sequence $\{f_n(x)p(x_n)\} \in \lambda$, $\mu \subset \Lambda(P)$. Consequently, the map $\Psi: X \to \mu$ defined by $\Psi(x) = \{f_n(x)\}$ is a bijective linear map. The seminorms generating the topology T_P are given by

$$Q_{p,\beta}(\alpha) = \sum_{n \geq 1} p(x_n) \ \beta_n |\alpha_n|$$

Therefore, by the fully λ-character of $\{x_n, f_n\}$, for every p in D_T and β in λ^x, $\beta > 0$, there exists a $q \in D_T$ such that

$$Q_{p,\beta}(\Psi(x)) \leq q(x)$$

On the other hand, for p in D_T,

$$p(\Psi^{-1}(\{f_n(x)\})) \leq \frac{1}{K_c} Q_{p,c}(\{f_n(x)\})$$

Hence $(X,T) \simeq (\mu, T_P/\mu)$. We next show that $\bar{\mu} = \Lambda(P)$. Let $\alpha \in \Lambda(P)$ but $\alpha \notin \bar{\mu}$. Then by the Hahn-Banach Theorem, there exists an $f \in (\Lambda(P))^*$ so that $\langle\alpha,f\rangle = 1$ and $\langle\beta,f\rangle = 0$ for each β in μ. The last equality yields $\langle e^n,f\rangle = 0$ for all $n \geq 1$. Thus $\langle\alpha,f\rangle = 0$, a contradiction and so $\bar{\mu} = \Lambda(P)$.

Finally, if (X,T) is sequentially complete, then $\mu = \bar{\mu}$. For, if $\mu \subsetneq \bar{\mu} = \Lambda(P)$, then we can find some α in $\Lambda(P)$ such that $\alpha \notin \mu$. Now

$$K_c \sum_{n\geq 1} |\alpha_n| p(x_n) \leq \sum_{n\geq 1} |\alpha_n| p(x_n) c_n < \infty, \ \forall\, p \in D_T.$$

Therefore $\sum_n \alpha_n x_n$ converges to x, say, in (X,T); that is, $\alpha = \{f_n(x)\} \in \mu$, a contradiction again. Hence $\mu = \Lambda(P)$.

<u>Proof of Theorem</u> 7.3.2: We prove this result in two parts. Where as in Part I we prove the fully ℓ^1-basis character of monomials, Part II exibits that the space $(H_{HY}(U), \tau_o)$ is complete and nuclear.

I, clearly, the set $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ is a countable subset of $H_{HY}(U)$.

Consider now an f in $H_{HY}(U)$ and a compact subset K of U. For $b \in U$ and $r \in \mathbb{N}$, define

$$[b]_r = \{s \in \omega : |s_i| \leq |b_i|,\ 1 \leq i \leq r \text{ and } s_i = 0,\ i > r\}.$$

As U is normal, $[b]_r$ is a finite dimensional polydisc in U. Therefore, for given $s \equiv \{s_i\}$ in K and $t \in [s]_r$, $r \in \mathbb{N}$, we have from Theorem 1.4.8, the following

$$f(t) = \sum_{m \in \mathbb{N}^r} a_m t^m$$

where

$$a_m = \frac{1}{(2\pi i)^r} \int \cdots \int_T \frac{f(u_1, \ldots, u_r, 0, 0, \ldots)}{u_1^{m_1+1} \cdots u_r^{m_r+1}} du_1, \ldots, du_r,$$

$$T = \{(u_1, \ldots, u_r) : |u_i| = |s_i|,\ i=1, \ldots, r\}.$$

Consequently, for $m \in \mathbb{N}^r$

(7.3.4) $$|a_m| \leq \frac{||f||_{[s]_r}}{|s^m|} \leq \frac{||f||_K}{|s^m|}$$

where for $A \subset U$, $||f||_A = \sup \{|f(x)| : x \in A\}$ and $|s^m| = |s_1^{m_1}| \ldots |s_r^{m_r}|$.

Since K is compact, by Proposition 7.2.2, we have a $\delta \equiv \{\delta_n\}$ with $\delta_n > 1$, $n \geq 1$ and $\{1/\delta_n\} \in \ell^1$ such that δK is a relatively compact subset of U. Applying (7.3.4) to δK we get

$$|a_m| \leq \frac{||f||_{\delta K}}{|(\delta s)^m|}, \forall m \in \mathbb{N}^r$$

$$\Longrightarrow \quad |a_m s^m| \leq \frac{||f||_{\delta K}}{\delta^m}, \forall m \in \mathbb{N}^r.$$

As the above inequality is true for each s in K, we get

(7.3.5) $\sup_{s\in K} |a_m s^m| \leq \dfrac{||f||_{\delta K}}{\delta^m}$, $\forall\, m \in \mathbb{N}^r$

(7.3.6) $\implies \sum_{m\in\mathbb{N}^{(\mathbb{N})}} \sup_{s\in K} |a_m s^m| \leq ||f||_{\delta K} \sum_{m\in\mathbb{N}^{(\mathbb{N})}} \dfrac{1}{\delta^m}$

Since $\mathbb{N}^r \subset \mathbb{N}^{r+1}$, $r \geq 1$ and $\{1/\delta_n\} \in \ell^1$, we have

$$\sum_{m\in\mathbb{N}^{(\mathbb{N})}} \frac{1}{\delta^m} = \prod_{n=1}^{\infty} \sum_{j=0}^{\infty} \left(\frac{1}{\delta_n}\right)^j = \prod_{n=1}^{\infty} \frac{1}{(1-\frac{1}{\delta_n})} = c$$

where c is a finite constant, as $\dfrac{1}{\prod_{n=1}^{\infty}(1-\frac{1}{\delta_n})}$ converges

for $1 - \dfrac{1}{\delta_n} \to 1$ as $n \to \infty$. Hence

(7.3.7) $\sum_{m\in\mathbb{N}^{(\mathbb{N})}} \sup_{s\in K} |a_m s^m| \leq c\, ||f||_{\delta K}$

Consequently, the series $\sum_{m\in\mathbb{N}^{(\mathbb{N})}} a_m s^m$ converges in the field $\mathbb{K}$ for each s in U. Thus, the function $\tilde{f}$ given by

(7.3.8) $\tilde{f}(s) = \sum_{m\in\mathbb{N}^{(\mathbb{N})}} a_m s^m$, $s \in U$

defines a hypoanalytic function on U, by (7.3.7).

Let $D = \bigcup_{r\geq 1} D_r$, where

$$D_r = \bigcup \{[s]_r : s \in U\}.$$

If $s \in U$, by our hypothesis $s^{(n)} \to s$ in $\beta(\lambda^x, \lambda)$. But $s^{(n)} \in [s]_n$ and so $s^{(n)} \in D$, $\forall\, n \geq 1$. Hence D is a dense subset of U. Observe that

$$\tilde{f}(s) = f(s), \quad \forall\, s \in D$$

Since both the functions f and $\tilde{f}$ are continuous on compact subsets of U, it follows that $f = \tilde{f}$ on U. Hence

$$(7.3.9) \qquad f(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m s^m \;, \forall\, s \in U$$

$$= \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m(s), \forall\, s \in U.$$

In order to show that $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ is a Schauder base for $(H_{HY}(U), \tau_o)$, it suffices to prove that the series $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m$ converges to f in the topology τ_o as the Schauder character of $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ is immediate from (7.3.5). Therefore, consider a compact subset K of U and $\varepsilon > 0$. Then for $\delta = \{\delta_n\}$ as above we can find a finite subset J_o of $\mathbb{N}^{(\mathbb{N})}$ such that

$$(7.3.10) \qquad \sum_{m \in \mathbb{N}^{(\mathbb{N})} \setminus J_o} \frac{1}{\delta^m} < \frac{\varepsilon}{\|f\|_{\delta K}}$$

Hence for any finite subset J of $\mathbb{N}^{(\mathbb{N})}$ with $J \supset J_o$, we have

$$(7.3.11) \qquad \|f - \sum_{m \in J} a_m f^m\|_K = \sup_{s \in K} \left| \sum_{m \in \mathbb{N}^{(\mathbb{N})} \setminus J_o} a_m s^m \right|$$

$$< \varepsilon$$

from (7.3.9), (7.3.5) and (7.3.10). Thus (7.3.11) yields the unordered convergence of the series $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m f^m$ to f in the topology τ_o.

The fully ℓ^1-character of the base $\{f^m : m \in \mathbb{N}^{(\mathbb{N})}\}$ is a consequence of (7.3.7) which can be written as

$$\sum_{m \in \mathbb{N}^{(\mathbb{N})}} \| a_m f^m \|_K \leq c \, \|f\|_{\delta K} < \infty .$$

II. Let us first prove the completeness of the space $(H_{HY}(U), \tau_o)$ and so consider a τ_o-Cauchy net $\{f_\alpha : \alpha \in I\}$ in $H_{HY}(U)$. Then

$$f_\alpha(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m^\alpha s^m , \; s \in U, \; \alpha \in I$$

where a_m^α's are uniquely determined scalars in the basis expansion of f_α's. Since the net $\{a_m^\alpha : \alpha \in I\}$ satisfies (7.3.7), it is a Cauchy net in the field $\mathbb{K}$ for each $m \in \mathbb{N}^{(\mathbb{N})}$. Hence there exists a set $\{a_m : m \in \mathbb{N}^{(\mathbb{N})}\}$ in $\mathbb{K}$ such that

(*) $$a_m = \lim_\alpha a_m^\alpha , \; m \in \mathbb{N}^{(\mathbb{N})} .$$

For given $\varepsilon > 0$ and a compact set K of U, let $\alpha_o \equiv \alpha_o(\varepsilon, K)$ in I be such that

$$\|f_\alpha - f_\beta\|_{\delta K} \leq \varepsilon, \; \alpha, \beta \geq \alpha_o$$

where $\delta \equiv \{\delta_n\}$ is the one as obtained in the Proposition 7.2.2.

Using (7.3.5) and (*), we get

$$|(a_m^\alpha - a_m) s^m| \leq \frac{\varepsilon}{\delta^m} , \; \alpha \geq \alpha_o$$

for s in K and $m \in \mathbb{N}^{(\mathbb{N})}$. Hence for s in K and $\alpha \geq \alpha_o$,

$$(+)\qquad \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m s^m| \leq \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |(a_m^{\alpha_o} - a_m) s^m| + \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m^{\alpha_o} s^m|$$

$$\leq cE + \sum_{m \in \mathbb{N}^{(\mathbb{N})}} |a_m^{\alpha_o} s^m|$$

$$< \infty$$

where

$$c = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{1}{\delta^m} = \frac{1}{\prod_{n=1}^{\infty} (1 - \frac{1}{\delta_n})} .$$

Consequently, we can define a function f on U as follows:

$$f(s) = \sum_{m \in \mathbb{N}^{(\mathbb{N})}} a_m s^m, \; s \in U$$

Further,

$$\sup_{s \in K} \left| \sum_{m \in \mathbb{N}^{(\mathbb{N})}} (a_m^{\alpha} - a_m) s^m \right| < \sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{\varepsilon}{\delta^m} = c\varepsilon, \; \alpha \geq \alpha_o,$$

yields that $f_\alpha \to f$ uniformly on K.
Since uniform limit of continuous function on compact sets is continuous on compact sets, f is in $H_{HY}(U)$ and it is the required τ_o-limit of $\{f_\alpha\}$.

For nuclearity, observe that the space $(H_{HY}(U), \tau_o)$ can be made topologically isomorphic to the Köthe sequence space $(\Lambda(P), T_P)$ by the Proposition 7.3.3, where

$$P = \{\{||f^m||_K\}_{m \in \mathbb{N}^{(\mathbb{N})}} : K \text{ varies over compact subsets of } U\}$$

Since from (7.3.5) we have

$$||f^m||_K \leq \frac{1}{\delta^m} ||f^m||_{\delta K}, \quad m \in \mathbb{N}^{(\mathbb{N})}$$

where $\sum_{m \in \mathbb{N}^{(\mathbb{N})}} \frac{1}{\delta^m} < \infty$ and δK is a relatively compact subset of U, the space $(H_{HY}(U), \tau_o)$ is nuclear by Proposition 1.3.8. This establishes the result completely.

REFERENCES

[1] Alexander, A. : Analytic functions on Banach spaces; Thesis, Univ. California, Berkeley, 1968.

[2] Allen, H.S. : Projective convergence and limit on sequence spaces; Proc. London Math. Soc. 48(1945), 310-338.

[3] Ansemil, J.M. : Topological duality on the function space $H(\mathbb{C}^n)$; Jour. Math. Anal. Appl. 67(1979), 188-197.

[4] Ansemil, J.M. and Ponte, S. : Example of a quasinormable Fréchet function space which is not a Schwartz space; Springer Verlag Lect. Notes in Math. 843 (1981), 1-9.

[5] Aron, R.M. : Holomorphy types for open subsets of a Banach space; Studia Math. 45(1973), 273-289.

[6] ———— : Compact polynomials and compact differentiable mappings between Banach spaces; Springer Verlag Lect. Notes in Math. 524 (1976), 46-59.

[7] Aron, R.M. and Cima, J.A. : A theorem on holomorphic functions on Banach spaces with basis; Proc. Amer. Math. Soc. 36(1972), 289-292.

[8] Arsove, M.G. : Proper bases and automorphisms in the space of entire functions; Proc. Amer. Math. Soc. 8(1957), 264-271.

[9] ———— : Proper bases and linear homomorphisms in the space of analytic functions; Math. Ann. 135 (1958), 235-243.

[10] Aurich, V. : The spectrum as envelop of holomorphy of a domain over arbitrary product of $\mathbb{C}$; Springer Verlag Lect. Notes in Maths. 364(1974), 109-122.

[11] Aytuna, A. and Terzioglu, T. : On certain subspaces of a nuclear power series of finite type; Studia Math. TXIX, (1980), 79-86.

[12] Barraso, J.A. : Topologies sur les espaces d'application holomorphes entre des espaces locatment convexes; C.R. Acad. Sci. Paris A-B271, (1970), A264-265.

[13] Barraso, J.A.; Matos, M.C. and Nachbin, L. : On bounded sets of holomorphic mappings; Springer Verlag Lect. Notes in Math. 364(1974), 91-108.

[14] Barraso, J.A. and Nachbin, L. : $\tau_0=\tau_\omega$ for domains in $\mathbb{C}^n$; MR 74h, # 46059, (1974).

[15] ______________ : Some topological properties of spaces of holomorphic mappings in infinitely many variables; Series of North Holland Math. Stud. 34 (1979), 67-91.

[16] Batty, C.J.K. : Nuclear faces of state spaces of C^*-algebra; Proc. Amer. Math. Soc. 86(2), (1982), 275-278.

[17] Bessaga, C.; Pelczynski, A. and Rolewicz, S. : On diametral approximative dimension and linear homogeneity of F-spaces; Bull. Acad. Polon. Sci. 9(1961), 677-683.

[18] Bianchini, M. : Silva holomorphy types, Borel transform and partial differential operators; Springer Verlag Lect. Notes in Math. 843 (1981), 41-67.

[19] Bierstedt, K.D. and Meise, R. : Nuclearity and Schwartz property in the theory of holomorphic functions on metrizable l.c. TVS; Series North Holland Math. Stud. 12(1977), 93-129.

[20] __________ : Aspects of inductive limits in space of germs of holomorphic function on locally convex spaces and application to a study of $(H(U),\tau_\omega)$; Series of North Holland Math. Stud. 340 , (1979), 111-178.

[21] Bierstedt, K.D.; Meise, R. and Summers , W.H. : Köthe set and Köthe sequence space Series North Holland Math. Stud. 71(1982), 27-91.

[22] Bjon, S. : On nuclear limit vector spaces; Springer Verlag Lect. Notes in Math. 915(1982), 27-39.

[23] Bochnak, J. : Analytic functions on Banach spaces; Studia Math. 35(1970), 273-292.

[24] Bochnak, J. and Siciak, J. : Separately analytic functions and envelop of holomorphy in $\mathbb{C}^n$; Springer Verlag Lect. Notes in Math. 71(1968), 101-132.

[25] ____________ : Remarks on domains of holomorphy in topological vector spaces; Bull. Acad. Polon. Sci. 19(1971), 9-12.

[26] ____________ : Polynomials and multilinear mappings in topological vector spaces; Studia Math. 39(2),(1971), 59-76.

[27] ____________ : Analytic functions on topological vector spaces, Studia Math. 29(1971), 77-112.

[28] Boland, P.J. : Some spaces of nuclearly holomorphic functions of bounded type; ATAS3a Quinzera de analyse functional e equacoes diferencials Parciais 2(1971), 175-189.

[29] ____________ : Some spaces of entire and nuclearly entire functions on a Banach space I; J. Reine Angew Math. 270 (1974), 38-60.

[30] ____________ : Some spaces of entire and nuclearly entire functions on a Banach space II; J. Reine Angew Math. II 271, (1974), 8-27.

[31] ____________ : Malgranges theorems for entire functions on nuclear spaces; Springer Verlag Lect. Notes in Math. 364(1974), 135-144.

[32] Boland, P.J. : Holomorphic functions on DFN-spaces; Springer Verlag Lect. Notes in Maths. 474(1975), 81-103.

[33] ______ : Holomorphic functions on nuclear spaces; Trans. Amer. Math. Soc. 209 , (1975), 275-281.

[34] ______ : An example of a nuclear space in infinite dimensional holomorphy ; Arkiv for Math. 15(1977), 87-91.

[35] ______ : Duality and spaces of holomorphic functions; Series North Holland Math. Stud. 12(1977), 131-138.

[36] ______ : Holomorphic functions on nuclear spaces; Universidad de Santiago de Compostela, Spain (1977), 1-61.

[37] ______ : Polynomials and multilinear forms on fully nuclear spaces; Seminaire Paul Kree (Equations aux derives partielles on dimension infinie) 4e annee 2, 1977/78, 7p, 201 to 207.

[38] Boland, P.J. and Dineen, S. : Convolution operators on G-holomorphic functions in infinite dimension; Trans. Amer. Math. Soc. 190(4), (1974), 208-217.

[39] ______ : Duality theory for spaces of germs and holomorphic functions on nuclear spaces; Series North Holland Math. Stud. 34(1975), 197-207.

[40] ______ : Holomorphic functions on fully nuclear spaces; Bull. Soc. Math. France 106 (1978), 311-336.

[41] ______ : Functions holomorphes surdes espaces pleinement nucleaires; C.R. Acad.Sci. Paris, t.286(1978), 1235-1237.

[42] ______ : Holomorphy on the spaces of distribution; Pacific J.Math.92(1), (1981), 27-34.

[43] Borgen, M.; Meise, R. and D. Vogt : Entire functions on nuclear sequence spaces, J. Reine Angew. Math. 322 (1981), 196-220.

[44] Borgen, M.; Meise, R. and D. Vogt : $\Lambda(\alpha)$-nuclearity in infinite dimensional holomorphy; Math. Nachr. 106(1982), 129-146.

[45] Bremermann, H.J. : The envelops of holomorphy of tube domains in infinite diminsional Banach spaces; Pacific J. Math. 10(1960), 1149-1153.

[46] ________ : Pseudo convex domains in linear topological spaces; MR 31c # 683(1968).

[47] Chae, S.B. : Holomorphic germs on Banach spaces; Ann. Inst. Fourier Grenable, 21(3), (1971), 107-141.

[48] Coeure, G. : Prolongment analytique; Springer Verlag Lect. Notes in Math. 116(1970), 128-132.

[49] ________ : Fonctionnelles analytiques sur certains espaces de Banach; Ann. Inst. Fourier Grenoble 21(1971), 15-21.

[50] Colombeau, J.F. : On various notions of infinite dimensional holomorphy; Springer Verlag Lect. Notes in Math. 364(1973), 145-149.

[51] Colombeau, J.F. and Matos, M.C. : On some spaces of entire functions defined on infinite dimensional spaces; Pacific J. Math. 96(1), (1981), 63-75.

[52] Colombeau, J.F.; Meise, R. and Perrot, B. : A density result in spaces of Silva holomorphic mappings; Pacific J. Math. 84(1), (1979), 35-42.

[53] Colombeau, J.F. and Mujica, J. : Existence of holomorphic mappings with prescribed asymptotic expansions at a given set of points in infinite dimension; Nonlinear Anal. 5(2), (1931), 149-156.

[54] ________ : Holomorphic and differentiable mappings of uniform bounded types MR 84h # 46 0.55, (1984).

[55] Colombeau, J.F. and Perrot, B. : The Fourier Borel transform in infinitely many dimensions and applications; Springer Verlag Lect. Notes in Math. 843(1961), 163-186.

[56] De Grande De Kimpe, N. : Criteria for nuclearity in terms of generalized, sequence spaces; Archder Math. 28, (1977), 644-651.

[57] Dineen, S. : Holomorphy types on a Banach space; Thesis, University of Maryland 1969 (Studia Math. 39(1971), 242-287).

[58] ____________ : Holomorphic functions on a Banach space, Bull. Amer. Math. Soc. 76(4), (1970), 883-886.

[59] ____________ : Runge domains in Banach spaces; Proc. Roy. Irish Acad. 71(7), (1971), 35-39.

[60] ____________ : Bounding subsets of a Banach space; Math. Ann. 192(1971), 61-70.

[61] ____________ : Unbounded holomorphic functions on a Banach space; J. London Math. Soc. 4(3), (1972), 106-116.

[62] ____________ : Holomorphically significant properties of topological vector spaces; Colloq. Interstat. C.N.R.S.Paris, 208(1972), 25-84.

[63] ____________ : Holomorphically complete locally convex space; Springer Verlag Lect. Notes in Math. 332(1973), 112-123.

[64] ____________ : Holomorphic functions on locally convex space I and locally convex topologies on H(U); Ann. Inst. Fourier Grenoble 23, (1973), 19-54.

[65] ____________ : Holomorphic functions and surjective limits; Springer Verlag Lect. Notes in Math. 364,(1974), 1-12.

[66] ____________ : Equivalent definitions of holomorphic functions in infinite dimensions; Springer Verlag Lect. Notes in Maths. 474(1975), 62-83.

[67] Dineen, S. : Surjective limits of locally convex spaces and their applications; Bull. Soc. Math. France 103(1975), 441-509.

[68] __________ : Holomorphic germs on compact subsets of locally convex spaces; Springer Verlag Lect. Notes in Math. 843(1978), 247-263.

[69] __________ : Holomorphic functions on strong dual of a Frechet-Montel space; Review of Campinas conference on infinite dimensional holomorphy and application, 1979, 11-17.

[70] __________ : Holomorphic functions on nuclear sequence spaces; Series North Holland Math. Stud. 38(1980), 42-58.

[71] __________ : Topological properties inherited by certain subspaces of holomorphic functions, Math. Anal. Appl. Advances in Math. Suppl Stud. 7(a) AP New York (1981).

[72] __________ : Anticommutative analytic forms on fully nuclear spaces; (Rev. R. Acad. Ciene. Exactas Fis. Nat. Spain 75(1981), 155-175) MR 84e,#46055,(1984).

[73] __________ : Complex Analysis in Locally Convex Spaces; MR 84c, # 16051 (1984) (North Holland publishing Company Amsterdam 1981).

[74] __________ : Analytic functions on fully nuclear spaces; Studia Math. 73(1), (1982), 11-32.

[75] Dineen, S: Meise, R and Vogt, D. : Characterization of a nuclear Frechet space in which every bounded set is pluripolar; C.R. Acad. Paris Ser. I. Math. 295(5), (1982), 385-388.

[76] Dineen, S. and Nachbin, L. : Entire functions of exponential type bounded on the real axis and Fourier transforms of distributions with bounded supports; MR 43e # 887, (1974).

[77] Dubinsky, Ed. : Infinite type power series sub spaces of finite type power series spaces; Isral J. Math. 15(1973), 257-281.

[78] ——————— : Infinite type power series subspaces of infinite type fourier series spaces; Isral J. Math. 20(1975), 359-368.

[79] ——————— : Structure of Nuclear Frechet Spaces; Springer Verlag Lect. Notes in Math. 720(1979).

[80] Dubinsky, E.D. and Ramanujan M.S. : $\Lambda_\infty(\alpha)$-bases and nuclear spaces; J. Ind. Math. Soc. 36(1972), 333-345.

[81] ——————— : On λ-nuclearity; Mem. Amer. Math. Soc. 128(1972).

[82] Dubinsky, Ed. and Retherford, J.R. : Schauder bases in Kothe sequence spaces; Trans. Amer. Math. Soc. 130(1968), 265-280.

[83] Duren, P.L. : Theory of H^p-spaces; Academic Press, New York, 1970.

[84] Dynin, A.S. and Mityagin, B.S. : Critereon for nuclearity in terms of approximative dimensions, Bull. Acad. Polon. Sci. 8(1960), 535-540.

[85] Dwyer, T.A.W. : Partial differential equations in Fischer-Fock spaces for the Hilbert Schmidt holomorphy type; Bull. Amer. Math. Soc. 77 (1971), 725-730.

[86] ——————— : Convolution equations for vector valued entire functions of nuclear bounded type; Trans. Amer. Math. Soc. 217(1976), 105-119.

[87] Fehr, E. and Jarchow, H. : Folgennaume als universelle generatoron fur Klassen lokal-konvexen raume; Manus. Math. 25 (1978), 209-240.

[88] Fenske, C. and Schock, E. : Uber die diametrale dimension von lokàlkonvexen Raumen; BMBW-GMD, Bonn, Nr. 10(1969), 13-22.

[89] Fenske, C. and Schock, E. : Nuclear space of maximal diametral dimension; Compo. Math. 26(1973), 303-338.

[90] Freeman, M. and Reise, H. : A compact set that is locally holomorphically convex but not holomorphically convex; Pacific J. Math. 48(1973), 77-81.

[91] Garling, D.J.H. : On symmetric sequence spaces; Proc. London Math. Soc. 16(3), (1966), 85-106.

[92] ________ : The β.and γ-duality of sequence spaces; Proc. Camb. Phil. Soc. 63(1), (1967), 963-981.

[93] ________ : On topological sequence spaces; Proc. Camb. Phil. Soc. 63(2), (1967), 997-1019.

[94] Gautam, S.K.S. : Bases in certain spaces of analytic Dirichlet transformation; Dissertation, I.I.T. Kanpur, 1973.

[95] Gelfand, I.M. and Shilov, G.E. : Generalized Functions Vol. 4; Academic Press, New York (1964).

[96] Greenfield, S.J. and Wallach, N.R. : The Hilbert ball and bi-ball are holomorphically inequivalent; Bull. Amer. Math. Soc. 77(1971), 261-263.

[97] ________ : Automorphism groups of bounded domains in Banach space; Trans. Amer. Math. Soc. 166(1972), 45-57.

[98] Gregory, D.A. : Vector valued sequence spaces; Dissertation, Univ. of Michigan Ann. Anbor (1967).

[99] Gross, F. : Generalised series and orders and types of entire functions of several complex variables; Trans. Amer. Math. Soc. 120(1), (1965), 124-144.

[100] Grothendieck, A. : Sur certains espaces de fonctions holomorphes I,II; J. Reine Angew Math. 192(1953), 35-64, 77-95.

[101] Grothendieck, A. : Produits tensoriels topologiques et espaces nucleaire; Mem. AMS 16(1955).

[102] Gupta, C.P. : Malgrange theorems for nuclearly entire functions of bounded type on a Banach space; Dissertation Univ. Rochester, New York (1966).

[103] ____________ : On the Malgranges theorems for nuclearly entire functions of bounded type on a Banach spaces; Notas de Matematica 37 I.M.P.A. Rio de Janeiro, Brasil (1968).

[104] ____________ : Convolution operators and holomorphic mappings on a Banach space; Sem. Anal. Mod. (2) Univ. Sherbrooke, Quebec, (1969).

[105] ____________ : On the Malgrange Theorem for nuclearly entire functions of bounded type on a Banach space; Indag Math. 32(1970), 356-358.

[106] Gupta, M. : A topological study of spaces of analytic functions of several variables; Dissertation, I.I.T. Kanpur (1973).

[107] Gupta, M.; Kamthan, P.K. and Patterson, J. : Duals of generalized sequence spaces; J. Math. Anal. Appl. 82(1), (1981), 152-168.

[108] Gupta, M.; Kamthan, P.K. and Rao, K.L.N. : The generalized sequence spaces $c, c_o, \ell^1, \ell^\infty$ and their Kothe duals; Tamkang Math.J.7(2) (1976), 175-178.

[109] ____________ : Duality in certain generalized sequence spaces; Bull. Inst. Math. Aca. Sinica, 5(2), (1977), 285-298.

[110] Harris, L.A. : The numerical range of holomorphic functions on Banach spaces; Amer. J. Math. 93 (1971), 1005-1019.

[111] ____________ : Bounds on the derivatives of holomorphic functions of vectors; Funct. Anal. Appl. 24(1), (1972), 145-163.

[112] Herve, M. : Analytic continuation on Banach Spaces; Springer Verlag Lect. Notes in Math. 185(1975), 63-75.

[113] Hille, E. and Philips, R.S. : Functional Analysis and Semi-Groups Providence, Rhode Island, Amer. Math. Soc. (1957).

[114] Hirschowitz, A. : Prolongement analytique en dimension infinie; C.R. Acad. Sci. Paris Ser A-B 270(1970), 403-413.

[115] ______________ : Bornologie des espaces de functions analytiques en dimension infinie; Springer Verlag Lect. Notes in Math. 205(1971), 21-33.

[116] ______________ : Sur le non-prolongement des varietes analytiques Banachique r'eelles; C.R. Acad. Sc. Paris 269(1972), 844-847.

[117] Hormander, L. : Introduction to Complex Analysis in Several Complex Variables; Von Nostrand, New York (1966).

[118] ______________ : Linear Partial Differential Operators; Springer Verlag Berlin (1963).

[119] Horvath, J. : Topological Vector Spaces and Distribution I; Addison-Wesley Don Mills, Ontario (1966).

[120] Husain, T. and Kamthan, P.K. : Spaces of entire functions represented by Dirichlet series; Collect. Math. 19 Fasc3^{o} Ano (1968), 203-216.

[121] Isidro, J.M. : On τ_δ-completeness of H(U); Rev. Math. Hisp. Amer. 39(4), (1979), 162-165.

[122] Iyer, V.G. : On spaces of integral functions; J. Indian Math. Soc. 12(2), (1948), 13-30.

[123] ______________ : The space of integral function II; Quart. J. Math. Oxford Ser. 2(1), (1950), 86-96.

[124] Iyer, V.G. : The space of integral function III; Proc. Amer. Math. Soc. 3(1952), 874-883.

[125] ____________ : On the space of integral functions IV Proc. Amer. Math. Soc. 4(1), (1956), 644-649.

[126] ____________ : On the space of integral functions V; J. Indian Math. Soc. 24(1), (1960), 269-278.

[127] Jarchow, H. : Barrelledness and Schwartz spaces; Math. Ann. 200(1973), 241-252.

[128] Jose, M. : On spaces of vector valued continuous function; Bull. Sci. Math. 107(2), (1983), 177-192.

[129] ____________ : Necessary and sufficient conditions for C(X,E) to be barrelled or infrabarrelled, Simon Stevin 57, (1983), 103-123.

[130] Kamthan, P.K. : FK-spaces of entire Dirichlet functions, Collect Math. 20(1969), 271-280.

[131] ____________ : A study on spaces of entire functions of several complex variables; Yokohama Math. J. 21(1), (1973), 17-20.

[132] ____________ : Bases in a Certain Class of Frechet spaces; Tamkang J. Math. 7(1), (1976), 41-49.

[133] ____________ : Kolmogorov's diameters and their applications; Aligarh Bull Math. 7(1977), 9-21.

[134] Kamthan P.K. and Deheri, G.M. : An extended class of analytic functions of several complex variables; Liberata Math. 4(1), (1984), 21-59.

[135] Kamthan, P.K. and Gupta, M. : Analytic functions in bi-cylinder; J. Indian Pour Appl. Math. 5(12), (1974), 1119-1126.

[136] ____________ : Expansion of entire functions of several complex variables; Trans. Amer. Math. Soc. 192(1974), 371-382.

[137] Kamthan, P.K. and Gupta, M. : Space of entire functions of several complex variables, finite order point; Math. Japonicae 20(1), (1975), 7-19.

[138] ______________ : Infinite matrices and tensorial transformations; Acta. Math. Viet. 23(2), (1976), 43-52.

[139] ______________ : Sequence Spaces and Series; Marcel Dekker Inc. New York (1981).

[140] ______________ : Theory of Bases and Cones; Pitman, Advanced Publishing Programme London (1985).

[141] ______________ : Theory of Bases-Behaviour and Stability; Under preparation.

[142] Kamthan, P.K. and Sofi, M.A. : λ-bases and λ-nuclearity; Math. Anal. Appl. 98(2), (1984), 001-010.

[143] Katsaras, A.K. : On the topologies τ_ω, τ_δ and τ_λ of Nachbin; Ann. Soc. Sci. Bruxelles Ser I, 95(1), (1981), 21-34.

[144] Kaup, L. and Kaup, B. : Holomorphic Functions of Several Variables; Walter de Gruyter Berlin, New York (1983).

[145] Kelley, J.L. : General Topology; East-West Press, (1968).

[146] Kelley, J.L. and Namioka, I. : Linear Topological Spaces; Von Nostrand, Princeton (1963).

[147] Kisliakov, S.V. : What is needed for a 0-absolute mapping to be nuclear; Springer Verlag Lect. Notes in Maths. 846 (1979), 336-364.

[148] Kolmogorov, A.N. : On some asymptotic characterization of totally bounded metric spaces; DAN U.S.S.R. 108(1956), 385-388.

[149] ______________ : On the linear dimension of topological vector spaces; DAN U.S.S.R. 120(1958), 239-241.

[150] Kolmogorov, A.N. and Tikhomirov, V.M. : ε-entropy and ε-capacity of sets in function spaces; Amer. Math. Soc. Transl. 12(2), (1961), 277-364.

[151] Komura, T. and Komura, Y. : Uber die Einbutting der nuklearin Raumen in (S) ; Math. Ann. 162 (1966), 284-288.

[152] Köthe, G. : Topological Vector Space I; Springer Verlag, Berlin-Heidelberg New York (1969).

[153] Köthe, G. and Toeplitz, O. : Lineare Raume mit unendlich vielen Koordinaten and Ringe unendlichen Matrizen; J. Reine Angew Math. 17 (1934), 193-226.

[154] ———— : On a class of conditional Köthe function space; Math. Ann. 160 (1965), 131-145.

[155] Kree, P. : Nuclear triplets and the kernel theorem in infinite dimensional analysis; C.R. Acad. Sci. Paris Ser A-13287 (8), A635-637 (1978).

[156] Krishnamurthy, V. : On the spaces of certain class of entire functions; Aust. Math. Soc. 1(2), (1960), 147-170.

[157] Kuhn, T. : On the γ-nuclearity of Gaussian covariances and the composition of nuclear operators; Math. Ann. 262, (198 (1983), 377-381.

[158] Kwapien, S. and Smolenski, W. : On the nuclearity of a dual space with the convergence in Probability topology; Z. Wahrsch. Verzo. Gebite 59(2), (1982), 197-201 (MR # 46069, 84e, (1984))

[159] Lance, E.C. : Tensor product and nuclear C*-algebra and application; MR # 46051, 84d (1984).

[160] Lelong, P. : Characterization of holomorphic functions on Banach spaces; Anals Math. 46(2), (1965), 585-593.

[161] Lelong, P. : Some results on holomorphic mappings and plurisubharmonic functions in topological vector spaces; Springer Verlag Lect. Notes in Math. 185(1971), 97-124.

[162] Maddox, I.J. : Spaces of strongly summable sequences; Quart. J. Math. Oxford 18(2) (1967), 345-355.

[163] ____________ : Elements of Functional Analysis; Cambridge University Press (1970).

[164] Malgrange, B. : Existence et approximation des solutions aux derives partielles et des equations des convolutions; Ann. Inst. Four. Grenoble 6(1955-56), 271-355.

[165] Markusevich, A.I.: Sur les bases dans l'espaces des functions analytiques; Amer. Math. Soc. Transl. 22(1962), 1-42.

[166] Martineau, A. : Sur les fonctionnellee analytiques et la transformations de Fourier-Borel, Jour. d'Analyse Mathematique 11(1963), 1-164.

[167] Matarasso, S. and Trisi, M. : Compactness theorems in unbounded domains; Bull. Un. Mat. Ital. 18(5), (1981), 517-537.

[168] Matos, M.C. : Holomorphic mappings and Domains of Holomorphy; Thesis, Univ. Rochester (1970).

[169] ____________ : The envelop of holomorphy of Riemann domains over a countable product of complex planes; Trans. Amer. Math. Soc. 167(1972), 379-387.

[170] ____________ : Domains of τ-holomorphy in a seperable Banach space; Math. Ann. 195(1972), 273-278.

[171] ____________ : Holomorphic maps in Baire topological vector spaces; Bull. Un Ital. 6(4), (1972), 67-70.

[172] Matos, M.C. and Nachbin, L. : Entire functions on locally convex spaces and convolution operators; MR # 46057, 79f (1979).

[173] ______ : Silva holomorphy types; Springer Verlag Lect. Notes in Math. 843, (1981), 437-487.

[174] Matyszczyk, C. : Approximation of analytic and continuous mappings by polynomials on Frechet Spaces; Studia Math. 60(3), (1977), 223-238.

[175] Meise, R. : Raume holomerpher Vektor Funktionen mit Wachstumsbeding ungen and topologische tensor produkte; Math. Ann. 199(1972), 293-312.

[176] ______ : A remark on the ported anal compact open topology on H(U); Proc. Royl. Irish. Acad. 82(1), (1982), 121-128.

[177] Meise, R. and Vogt. D. : Structure of spaces of holomorphic functions on infinite dimensional polydiscs; Studia Math. T. LXXV, (1983), 235-252.

[178] ______ : An interpolation of τ_ω and τ_δ as normal topologies of sequence spaces; MR 84h,# 46053(1984).

[179] Mityagin, B.S. : The connection between the ε-entropy, the rapidity of approximation and the nuclearity of a compactum in a linear space; DAN U.S.S.R. 134(1960), 765-768.

[180] ______ : Approximative dimension and Bases in nuclear spaces; Russian Math. Surveys 16(1961), 59-127.

[181] Mityagin, B.S. and Pelczynski, A. : On the non existence of linear isomerphisms between Banach spaces of analytic functions of several complex variables; Studia Math. 56(1975), 85-96.

[182] Moraes, L.A. : Theorems of Cartan -Thullen type and Runge domains; Series of North-Holland Math. Stud. 34(1979), 521-561.

[183] Moraes, L.A. : Holomorphic functions on strict inductive limit spaces; Result Math. 4(2), (1981), 201-212.

[184] ________ : An example of $\lambda(P)$-nuclear Frechet space; J. Reine Angew Math. 301 (1978), 289-294.

[185] Moscatelli, V.B. : New examples of nuclear Frechet spaces without basis; Series North Holland Math. Stud. 71(1982), 112-142.

[186] Mujica, J. : Gemenes holomorfosy funciones holomorfas en espacios de Frechet, Universidad de Santiago de Compostelo, Spain (1978).

[187] ________ : Domains of holomorphy in (DFC)-spaces; Springer Verlag Lect. Notes in Math. 843(1978), 488-499.

[188] Nachbin, L. : Topological vector space of continuous functions; Proc. Nat. Acad. Sci. USA 40(1954), 51-58.

[189] ________ : Lectures on the theory of distributions; Univ. Rochester, USA (1963).

[190] ________ : On the topology of the space of holomorphic functions on a given open set; Indag Math. 70(1967), 13-37.

[191] ________ : Topology on the Space of Holomorphic Mappings; Springer Verlag (1969).

[192] ________ : Holomorphic Functions;Domains of Holomorphy and Local Properties; Series North Holland Math. Stud. (1970).

[193] ________ : Convolution operators in spaces of nuclearly entire functions on a Banach space; Proc. Symp. Funct. Anal. Red. Fiel. Univ. Chicago, (1970).

[194] ________ : Concerning holomorphy types for Banach spaces; Studia Math. 38(1970), 407-412.

[195] Nachbin, L. : Recent developments in infinite dimensional holomorphy; Bull. Amer. Math. Soc. 79(1973), 615-640.

[196] ____________ : On vector valued versus scalar valued holomorphic continuation; Indag Math. 35(1973), 352-374.

[197] ____________ : A glimpse at infinite dimensional holomorphy; Springer Verlag Lect. Notes in Math. 364(1974), 123-137.

[198] ____________ : Why holomorphy in infinite dimensions; L. Enseignent Math. 26(1980), 257-269.

[199] Nachbin, L. and Dineen, S. : Entire functions of exponential type bounded on the real axis and Fourier transforms of distributions with bounded supports; Isral J. Math. 13(1973), 321-326.

[200] Nelimarkka, E. : On spaces of holomorphic functions on locally convex spaces defined by an operator ideal; Math. Gazz. # 846, (1983).

[201] Noverraz, P. : Rings of analytic functions; Springer Verlag Lect. Notes in Math. 71(1968), 119-134.

[202] ____________ : Fonctions analytiques et theorems de prolongement dans les espaces vectoriels topologiques complexes; Springer Verlag Lect. Notes on Math. 116(1970), 21-29.

[203] ____________ : Sur les applications analytiques en dimension infinie; C.R. Acad. Sci. Paris Ser. A-B274 (1972), A760-A762.

[204] ____________ : Approximation of holomorphic and plurisubharonomic functions in certain Banach spaces; Springer Verlag Lect. Notes in Math. 364 (1974), 61-73.

[205] Nyberg, K. : Tameness of pairs of nuclear power series spaces and related topics; Trans. Amer, Math. Soc. 2(283),(1984), 645-660.

[206] O'Brien, D.M. : A simple test for nuclearity of integral operators on $L_2(\mathbb{R}^n)$; J. Aust. Math. Soc. Ser. A33(1982), 193-196.

[207] Pelczynski, A. : On the approximation of S-spaces by finite dimensional spaces; Bull. Acad. Polon. Sci. 5(9), (1957), 879-881.

[208] Persson, A. : On a class of conditional Köthe function spaces; Math. Ann. 160(1) (1965), 131-145.

[209] Pietsch, A. : Nuclear Locally Convex Spaces; Springer Verlag (1972).

[210] ____________ : Operator Ideals; North-Holland Publishing Company (1980).

[211] Pincuk, S.I. : Analytic continuation of holomorphic mappings on $\mathbb{C}^I$; MR 79f, # 46055,(1979).

[212] Pombo, D.J.P. : Semi-reflexivity of the space of continuous m-homogeneous polynomial; An. Acad. Bras. Cienc. 55(1983), 11-14.

[213] Porcelli, P. : Linear Spaces of Analytic Functions; Rand McNally, Chicago, (1966).

[214] Ramanujan, M.S. : On λ-nuclearity; Mem. Amer. Math. Soc. 128(1972).

[215] Ramanujan, M.S. and Rosenberger, B. : On $\lambda(P,\varphi)$-nuclearity; Comp. Math. 34(2), (1977), 113-125.

[216] Ramanujan, M.S. and Terzioglu, T. : Diametral dimensions of cartesian products, stability of smooth sequence spaces and applications; J. Für.Math. 280(22), (1973), 163-171.

[217] ____________ : Power series spaces $\Lambda_k(\alpha)$ of finite type and related nuclearities; Studia Math. 53(1975), 1-13.

[218] ____________ : Subspaces of smooth sequence spaces; Studia Math. 115(1979), 299-311.

[219] Ramanujan, M.S. and Terzioglu, T. : Compact and nuclear maps on power series spaces; Math. Nachr. 93(1979), 159-164.

[220] Randtke, D.J. : A simple example of a universal Schwartz space; Proc. Amer. Math. Soc. 37(1973), 185-188.

[221] ______________ : Structure theorem for Schwartz Spaces; Math. Ann. 201 (1973), 171-176.

[222] Rickart, C.E. : Analytic functions of an infinite number of variables; Duke Math. J. 36(1969), 581-597.

[223] ______________ : Plurisubharmonic functions and convexity properties for the general function algebras; Trans. Amer. Math. Soc. 169(1972), 1-24.

[224] ______________ : Holomorphic extensions and domains of holomorphy for the general function algebras; Springer Verlag Lect. Notes in Math. 364 (1973), 80-91.

[225] Robertson, A.P. and Robertson, W.J. : Topological Vector Spaces; Cambridge University Press, Cambridge (1970).

[226] Robinson, W.B. : On $\Lambda_1(\alpha)$-nuclearety; Duke Math. J. 40(3), (1973), 541-546.

[227] ______________ : Relationship between λ-nuclearity and pseudo-μ-nuclearity; Trans. Amer. Math. Soc. 201(1975), 291-303.

[228] Rolewicz, S. : On spaces of holomorphic functions; Studia Math. 21(1961), 135-160.

[229] ______________ : On representation of the spaces of holomorphic functions by the matrix spaces; MR 80g, # 46060 (1980).

[230] Rosenberger, B. : φ-nukleare Raum; Math. Nachr. 52 (1972), 147-160.

[231] Ruckle, W.H. : Topologies on sequence spaces; Pacific J. Math. 42(1), (1972), 235-249.

[232] Ruckle, W.H. : Sequence Space; Pitman Advanced Publishing Program (1981).

[233] Rudin, W. : Functional Analysis; Tata McGraw Hill Publishing Company, New Delhi (1974).

[234] Saxon, S.A. : Embedding nuclear spaces in products of an arbitrary Banach space; Proc. Amer. Math. Soc. 34(1972), 138-140.

[235] Schaefer, H.H. : Topological Vector Spaces; Springer Verlag (1971).

[236] Schatten, R. : The cross space of linear transformations I; Ann. Math. 47(1946), 73-84.

[237] Schatten, R. and Neumann, J. Von. : The cross space of linear transformations II; III; Ann. Math. 47(1946), 608-630, 49(1948), 557-582.

[238] Schock, E. : Diametrale dimension, approximative dimension and angwendungen; BMBW G.M.D. 43(1971), 48-73.

[239] Schottenloher, M. : Bounding sets in Banach space and regular classes of analytic functions; Springer Verlag Lect. Notes in Math. 384 (1974), 138-149.

[240] ______ : Spectrum and envelop of holomorphy for infinite dimensional Riemann Domain; Math. Ann. 263(2), (1983), 213-219.

[241] Schwartz, L. : Theorie des distributions, tomes I and II; Hermann, Paris, (1950), (1951).

[242] ______ : Theories des noyarex; Proc. Int. congr. Math. 1(1950), 220-230.

[243] Siciak, J. : A polynomial lemma and analytic mappings in topological vector spaces; Springer Verlag Lect. Notes in Math. 275(1972), 131-142.

[244] Silvano, M. and Trisi, M. : Compactness theorems in the unbounded domains; Bull. un. Ital. Mat. B18(5), (1981), 517-537.

[245] Sofi, M.A. : λ-base and λ-nuclearity; Dissertation I.I.T. Kanpur (1981).

[246] Spuhler, P. : Λ-nukleare Raume; Dissertation Univ. Frankfurt (1970).

[247] Stevenson's, J.O. : Holomorphy of compositions; MR 79 e , # 46057(1979).

[248] Synowiec, J. : Distribution, the evolution of a mathematical theory; Hist. Math. 10(1983), 149-183.

[249] Taylor, B.A. : Some locally convex spaces of entire functions; Proc. Symp. Pure Math. AMS(1968).

[250] Terzioglu, T. : On Schwartz spaces; Math. Ann. 182(1969), 236-242.

[251] ______________ : Die diametrale dimension von lokalkonvexn raumen ; Collect. Math. 20(1), (1969), 49-99.

[252] ______________ : Smooth sequence spaces and associated nuclearity; Proc.Amer. Math. Soc. 37(2), (1973), 497-504.

[253] ______________ : Stability of smooth sequence spaces, J. Reine Angew Math. 276(1975), 184-189.

[254] Titchmarsh, E.C. : Theory of Functions; Oxford, (1939).

[255] Treves, J.F. : Lectures on linear partial differential equations with constant coefficients; Institute de Mat. Pure Appl. Rio de Janeiro (1961).

[256] ______________ : Topological Vector Spaces, Distributions and Kernels, AP(1967).

[257] Urysohn, P. : Sur un probleme de M. Frechet relatif aux classess des fonction holomorphes; C.R. Congr. Soc. Sav Sci. (1924).

[258] Vigue, J.P. and Isidro, J.M. : Topology of the group of analytic automerphisms of a bounded circular domain; MR 84n, # 46058, (1984).

[259] Vogt, D. : Characterization of the subspace of nuclear stable power series spaces of finite type; Studia Math. 71(3), (1981-82), 251-270.

[260] ____________ : An isomerphism theorem for power series spaces; Arch. Math. 38(6), (1982), 540-548.

[261] Waelbroeck, L. : Nuclearity of $\theta(U)$; Series North-Holland Math. Stud. 12(1977), 425-435.

[262) Weill, G.G. : A note on monomials on several complex variables; Proc. Amer. Math. Soc. 42(1974), 541-543.

[263] Wilansky, A. : Functional Analysis; Blausdell' Publishing Company USA (1964).

[264] ____________ : Modern Methods in Topological Vector Spaces; McGraw-Hill Company (1978).

[265] Wong, Y.C. : Schwartz Spaces; Nuclear Spaces and Tensor Products; Springer Verlag Lect. Notes on Math. 726(1979).

[266] Zaine, M.C.F. : Envelops of Silva holomorphy; Springer Verlag Lect. Notes in Math. 843(1978), 581-614.

[267] Zalote, N.M. : Theorem of Phillips in locally convex topological vector spaces; MR 74g , # 46057, (1974).

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